UNIT
4

Factorization and Algebraic Manipulation

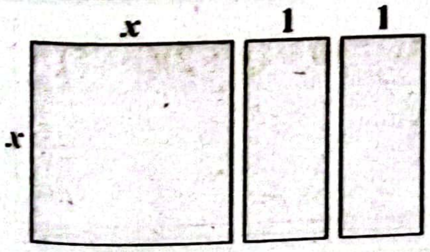
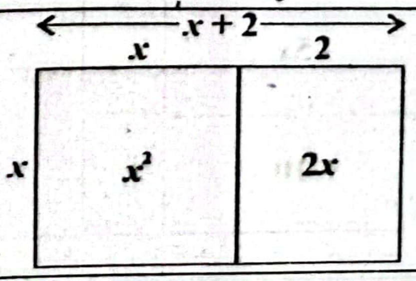
Students' learning outcomes

At the end of the unit, the students will be able to:

- Identify common factors, trinomial factoring, concretely, pictorially and symbolically.
- Factorize quadratic and cubic algebraic expressions:
  - $a^4+a^2b^2+b^4$ or a^4+b^4
  - x^2+px+q
  - ax^2+bx+c
  - $(ax^2+bx+c)(ax^2+bx+d)+k$
  - $(x+a)(x+b)(x+c)(x+d)+k$
  - $(x+a)(x+b)(x+c)(x+d)+kx^2$
  - $a^3+3a^2b+3ab^2+b^3$
  - $a^3+3a^2b+3ab^2+b^3$
  - $a^3-3a^2b+3ab^2-b^3$
  - $a^3 \pm b^3$
- Find highest common factor and least common multiple of algebraic expressions and know relationship of LCM and HCF.
- Find square root of algebraic expression by factorization and division.
- Apply the concepts of factorization of quadratic and cubic algebraic expression to real-world problems (such as engineering, physics, and finance.)

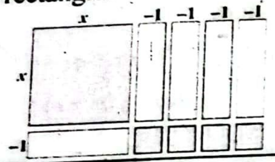
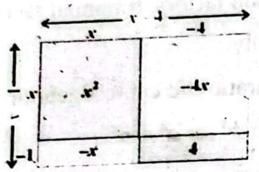
Example 1: Find common factor of x^2+2x concretely, pictorially and symbolically.

Solution: We arrange one x^2 tile and two x tiles into a rectangle.

Concretely	Pictorially	Symbolically
		$x^2+2x=x(x+2)$

Example 2: Factorize $x^2 - 5x + 4$ concretely, pictorially and symbolically.

Solution:

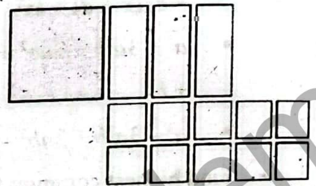
Concretely	Pictorially	Symbolically
We arrange one x^2 tile, five $-x$ tiles and four unit tiles into a rectangle. 		x^2-5x+4 $=(x-1)(x-4)$

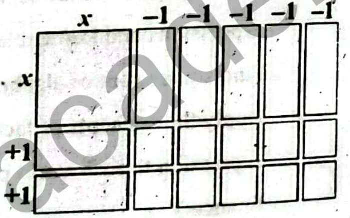
Example 3: Factorize $x^2 - 3x - 10$ concretely, pictorially and symbolically.

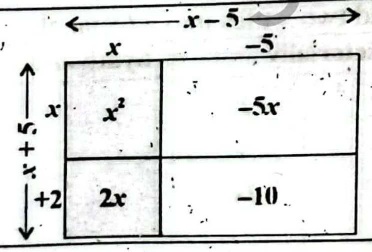
Solution:

Concretely we arrange one x^2 tile, three $-x$ tiles and ten -1 tiles into rectangle.

We see that there are not enough rectangular tiles to make a larger rectangle. To fix this issue, we add zero pair. Adding two x tiles and two $-x$ tiles does not change the given expression because $2x - 2x = 0$.





Pictorially	Symbolically
	$x^2 - 3x - 10 = (x + 2)(x - 5)$

Example 4: Factorize: $x^2 + 9x + 14$

Product of factors	Sums of factors
14 × 1 = 14	14 + 1 = 15
17 × 2 = 14	17 + 2 = 91

Solution: Two numbers whose product is +14 and their sum is 9 are +2, +7.

So, $x^2 + 9x + 14$

$= x^2 + 2x + 7x + 14$

$= x(x + 2) + 7(x + 2)$

$= (x + 2)(x + 7)$

Example 5: Factorize: $x^2 - 11x + 24$

Solution: Two numbers whose product is +24 and their sum is -11 are $-8, -3$.

So, $x^2 - 11x + 24$	Product of factors	Sums of factors
$= x^2 - 8x - 3x + 24$	24 × 1 = 24	24 + 1 = 25
$= x(x-8) - 3(x-8)$	18 × 3 = 24	18 + 3 = 11
$= (x-8)(x-3)$	(–8) × (–3) = 24	–8 – 3 = –11
	16 × 4 = 24	16 + 4 = 10
	112 × 2 = 24	12 + 2 = 14

Example 6: Factorize: $p^2 + 11p + 18$

Solution: $p^2 + 11p + 18$

$= p^2 + 9p + 2p + 18$ $\quad \because 9 + 2 = 11\ ,\ 9 \times 2 = 18$

$= p(p + 9) + 2(p + 9)$

$= (p + 9)(p + 2)$

In all quadratic trinomials factorized so far, the coefficient of x^2 was 1. We will now consider cases where the coefficient of x^2 is not 1.

Example 7: Factorize: $2x^2 + 7x + 26$

Solution:

Step – I: Multiply the coefficient of x^2 with constant term. i.e.,

> **Remember!**
> An expression having degree 2 is called a quadratic expression.

Step – II: List all the factors of 52:

1, 52	–1, – 52
2, 26	–2, – 26
4, 13	–4, –13

Step – III: Sum of factors equals middle term (17)

$1 + 52 = 53$	$-1 - 52 = -53$
$2 + 26 = 28$	$-2 - 26 = -28$
$\boxed{4 + 13 = 17}$	$-4, -13 = -17$

Step – IV: Change the middle term in the given expression

$$2x^2 + 17x + 26$$
$$= 2x^2 + 4x + 13x + 26$$

Step – V: Take out commons from first two terms and last two terms

$$= 2x(x + 2) + 13(x + 2)$$

Step – VI: Again, take common from both terms

$$= (x + 2)(2x + 13)$$

Example 8: Factorize: $3x^2 - 4x - 4$

Solution: $3x^2 - 4x - 4$

$$= 3x^2 + 2x - 6x - 4$$
$$\because 2 \times (-6) = -12\ ,\ +2 - 6 = -4$$
$$= x(3x + 2) - 2(3x + 2)$$
$$= (3x + 2)(x - 2)$$

EXERCISE 4.1

1. Factorize by identifying common factors.

(i)	$6x + 12$	(ii)	$15y^2 + 20y$
(iii)	$-12x^2 - 3x$	(iv)	$4a^2b + 8ab^2$
(v)	$xy - 3x^2 + 2x$	(vi)	$3a^2b - 9ab^2 + 15ab$

Solution: **Factorization by identifying common factors:**

(i) $6x + 12$:

Factor out the common factor 6:

$$6x + 12 = 6(x + 2)$$

(ii) $15y^2 + 20y$:

Factor out the common factor $5y$:

$$15y^2 + 20y = 5y(3y + 4)$$

(iii) $-12x^2 - 3x$:

Factor out the common factor $-3x$:

$$-12x^2 - 3x = -3x(4x + 1)$$

(iv) $4a^2b + 8ab^2$:

Factor out the common factor $4ab$:

$$4a^2b + 8ab^2 = 4ab(a + 2b)$$

(v) $xy^2 - 3xy + 2x$

Factor out the common factor x:

$$xy^2 - 3xy + 2x = x(y^2 - 3y + 2)$$

Further factorize if possible:

$$x(y^2 - 3y + 2) = x(y - 1)(y - 2)$$

(vi) $3a^2b - 9ab^2 + 15ab$:

Factor out the common factor $3ab$:

$$3a^2b - 9ab^2 + 15ab = 3ab(a - 3b + 5)$$

2. Factorize and represent pictorially:

(i)	$5x + 15$	(ii)	$x^2 + 4x + 3$
(iii)	$x^2 + 6x + 8$	(iv)	$x^2 + 4x + 4$

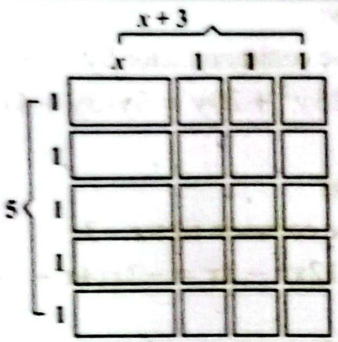
Solution: **Factorization:**

(i) $5x + 15$:

Factor out the common factor 5:

$$5x + 15 = 5(x + 3)$$

Pictorial Representation:



(ii) $x^2 + 4x + 3$:

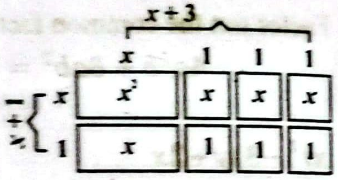
$x^2 + 4x + 3$

$= x^2 + 3x + x + 3$

$= x(x + 3) + 1(x + 3)$

$= (x + 1)(x + 3)$

Pictorial Representation:



(iii) $x^2 + 6x + 8$:

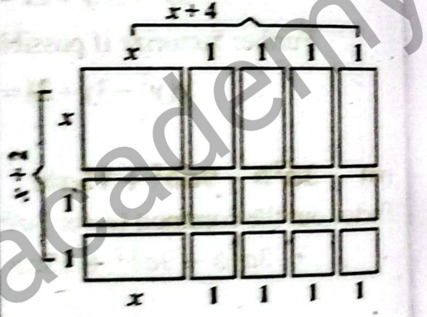
$x^2 + 6x + 8$

$= x^2 + 2x + 4x + 8$

$= x(x + 2) + 4(x + 2)$

$= (x + 2)(x + 4)$

Pictorial Representation:



(iv) $x^2 + 4x + 4$:

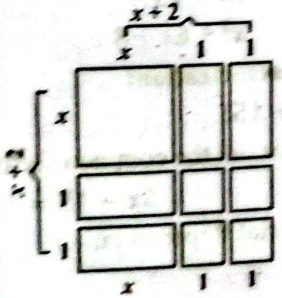
$x^2 + 4x + 4$

$= x^2 + 2x + 2x + 4$

$= x(x + 2) + 2(x + 2)$

$= (x + 2)^2$

Pictorial Representation:



3. **Factorize:**

(i) $x^2 + x - 12$ (ii) $x^2 + 7x + 10$

(iii) $x^2 - 6x + 8$ (iv) $x^2 - x - 56$

(v) $x^2 - 10x - 24$ (vi) $y^2 + 4y - 12$

(vii) $y^2 + 13y + 36$ (viii) $x^2 - x - 2$

Solution: **Factorization:**

(i) $x^2 + x - 12$:

$x^2 + x - 12$

$= x^2 - 3x + 4x - 12$

$= x(x - 3) + 4(x - 3)$

$= (x + 4)(x - 3)$

(ii) $x^2 + 7x + 10$:

$x^2 + 7x + 10$

$= x^2 + 5x + 2x + 10$

$= x(x + 5) + 2(x + 5)$

$= (x + 2)(x + 5)$

(iii) $x^2 - 6x + 8$:

$x^2 - 6x + 8$

$= x^2 - 4x - 2x + 8$

$= x(x - 4) - 2(x - 4)$

$= (x - 4)(x - 2)$

(iv) $x^2 - x - 56$:

$x^2 - x - 56$

$= x^2 - 8x + 7x - 56$

$= x(x - 8) + 7(x - 8)$

$= (x - 8)(x + 7)$

(v) $x^2 - 10x - 24$:

$x^2 - 10x - 24$

$= x^2 - 12x + 2x - 24$

$= x(x - 12) +2 (x - 12)$

$= (x - 12)(x + 2)$

(vi) $y^2 + 4y - 12$:

$y^2 + 4y - 12$

$= y^2 + 6y - 2y - 12$

$= y(y + 6) - 2(y + 6)$

$= (y + 6)(y - 2)$

(vii) $y^2 + 13y + 36$:

$y^2 + 13y + 36$

$= y^2 + 9y + 4y + 36$

$= y(y + 9) + 4(y + 9)$

$= (y + 9)(y + 4)$

(viii) $x^2 - x - 2$:

$x^2 - x - 2$

$= x^2 - 2x + x - 2$

$= x(x - 2) + 1(x - 2)$

$= (x - 2)(x + 1)$

4. **Factorize:**

(i) $2x^2 + 7x + 3$ (ii) $2x^2 + 11x + 15$

(iii) $4x^2 + 13x + 3$ (iv) $3x^2 + 5x + 2$

(v) $3y^2 - 11y + 6$ (vi) $2y^2 - 5y + 2$

(vii) $4z^2 - 11z + 6$ (viii) $6 + 7x - 3x^2$

Solution: **Factorization:**

(i) $2x^2 + 7x + 3$:

$2x^2 + 7x + 3$

$= 2x^2 + 6x + x + 3$
$= 2x(x + 3) + 1(x + 3)$
$= (2x + 1)(x + 3)$

(ii) $\mathbf{2x^2 + 11x + 15:}$

$2x^2 + 11x + 15$
$= 2x^2 + 6x + 5x + 15$
$= 2x(x + 3) + 5(x + 3)$
$= (2x + 5)(x + 3)$

(iii) $\mathbf{4x^2 + 13x + 3:}$

$4x^2 + 13x + 3$
$= 4x^2 + 12x + x + 3$
$= 4x(x + 3) + 1(x + 3)$
$= (4x + 1)(x + 3)$

(iv) $\mathbf{3x^2 + 5x + 2:}$

$3x^2 + 5x + 2$
$= 3x^2 + 3x + 2x + 2$
$= 3x(x + 1) + 2(x + 1)$
$= (3x + 2)(x + 1)$

(v) $\mathbf{3y^2 - 11y + 6:}$

$3y^2 - 11y + 6$
$= 3y^2 - 9y - 2y + 6$
$= 3y(y - 3) - 2(y - 3)$
$= (3y - 2)(y - 3)$

(vi) $\mathbf{2y^2 - 5y + 2:}$

$2y^2 - 5y + 2$
$= 2y^2 - 4y - y + 2$
$= 2y(y - 2) - 1(y - 2)$
$= (2y - 1)(y - 2)$

(vii) $\mathbf{4z^2 - 11z + 6:}$

$4z^2 - 11z + 6$
$= 4z^2 - 8z - 3z + 6$
$= 4z(z - 2) - 3(z - 2)$
$= (4z - 3)(z - 2)$

(viii) $\mathbf{6 + 7x - 3x^2:}$

$6 + 7x - 3x^2$
$= 6 - 2x + 9x - 3x^2$
$= 2(3 - x) + 3x\,(3 - x)$
$= (3 - x)\,(2 + 3x)$

Type – II: Factorization of the expression of the types $\mathbf{a^4 + a^2b^2 + b^4}$ **or** $\mathbf{a^4 + b^4}$

Let's factorize $a^4 + a^2b^2 + b^4$

$a^4 + a^2b^2 + b^4$
$= a^4 + b^4 + a^2b^2$
$= (a^2)^2 + (b^2)^2 + a^2b^2$
$= (a^2)^2 + (b^2)^2 + a^2b^2$
$= (a^2)^2 + (b^2)^2 + 2a^2b^2 - a^2b^2$

(Adding and subtracting a^2b^2)

$= (a^2 + b^2)^2 - a^2b^2$
$= (a^2 + b^2)^2 - (ab)^2$
$= (a^2 + b^2 - ab)(a^2 + b^2 + ab)$
$= (a^2 - ab + b^2)(a^2 + ab + b^2)$

Example 9: Factorize: $x^4 + x^2 + 25$

Solution: $x^4 + x^2 + 25$

$= x^4 + 25 + x^2$
$= (x^2)^2 + (5)^2 + 2(x^2)(5) - 2(x^2)(5) + x^2$

(Adding and subtracting $2(x^2)(5)$)

$= (x^2 + 5)^2 - 10x^2 + x^2$
$= (x^2 + 5)^2 - 9x^2$
$= (x^2 + 5)^2 - (3x)^2$
$= (x^2 + 5 - 3x)(x^2 + 5 + 3x)$
$= (x^2 - 3x + 5)(x^2 + 3x + 5)$

Example 10: Factorize: $x^4 + y^4$

Solution: $x^4 + y^4$

$= (x^2)^2 + (y^2)^2$

$= (x^2)^2 + (y^2)^2 + 2(x^2)(y^2) - 2(x^2)(y^2)$

(Adding and subtracting $2x^2y^2$)

$= (x^2 + y^2)^2 - (\sqrt{2}xy)^2$

$= (x^2 + y^2 - \sqrt{2}\,xy)(x^2 + y^2 + \sqrt{2}\,xy)$

$= (x^2 - \sqrt{2}\,xy + y^2)(x^2 + \sqrt{2}\,xy + y^2)$

Example 11: Factorize: $a^4 + 64$

Solution: $a^4 + 64$

$= (a^2)^2 + (8)^2$

$= (a^2)^2 + (8)^2 + 2(a^2)(8) - 2(a^2)(8)$

(Adding and subtracting $2(a^2)(8)$)

$= (a^2 + 8)^2 - 16a^2$

$= (a^2 + 8)^2 - (4a)^2$

$= (a^2 + 8 - 4a)(a^2 + 8 + 4a)$

$= (a^2 - 4a + 8)(a^2 + 4a + 8)$

ype – III: Factorization of the expression of the types

- $(ax^2 + bx + c)(ax^2 + bx + d) + k$
- $(x + a)(x + b)(x + c)(x + d) + k$
- $(x + a)(x + b)(x + c)(x + d) + kx^2$

For explanation consider the following examples:

Example 12: Factorize: $(x^2 + 5x + 4)(x^2 + 5x + 6) - 3$

Solution: $(x^2 + 5x + 4)(x^2 + 5x + 6) - 3$

$= (y + 4)(y + 6) - 3$ (Let $y = x^2 + 5x$)

$= y^2 + 6y + 4y + 24 - 3$

$= y^2 + 10y + 21$

$= y^2 + 7y + 3y + 21$

$= y(y + 7) + 3(y + 7)$

$= (y + 7)(y + 3)$

$= (x^2 + 5x + 7)(x^2 + 5x + 3)$ $(\because y = x^2 + 5x)$

Example 13: Factorize: $(x + 2)(x + 3)(x + 4)(x + 5) - 15$

Solution: $(x + 2)(x + 3)(x + 4)(x + 5) - 15$

Re-arrange the given expression because $2 + 5 = 3 + 4$

$[(x + 2)(x + 5)][(x + 3)(x + 4)] - 15$

$= (x^2 + 5x + 2x + 10)\ (x^2 + 4x + 3x + 12) - 15$

$= (x^2 + 7x + 10)(x^2 + 7x + 12) - 15$

$= (y + 10)(y + 12) - 15$ Let $y = x^2 + 7x$

$= y^2 + 12y + 10y + 120 - 15$

$= y^2 + 22y + 105$

$= y^2 + 15y + 7y + 105$

$= y(y + 15) + 7(y + 15)$

$= (y + 15)(y + 7)$

$= (x^2 + 7x + 15)(x^2 + 7x + 7)$ $(\because y = x^2 + 7x)$

Example 14: Factorize: $(x - 2)(x + 2)(x + 1)(x - 4) + 2x^2$

Solution: $(x - 2)(x + 2)(x + 1)(x - 4) + 2x^2$

$= [(x - 2)(x + 2)][(x + 1)(x - 4)] + 2x^2$

$[\because (-2) \times 2 = 1 \times (-4)]$

$= (x^2 - 2^2)(x^2 - 4x + x - 4) + 2x^2$

$= (x^2 - 4)(x^2 - 3x - 4) + 2x^2$

$= y(y - 3x) + 2x^2$ Let $y = x^2 - 4$

$= y^2 - 3xy + 2x^2$

$= y^2 - 2xy - xy + 2x^2$

$= y(y - 2x) - x(y - 2x)$

$= (y - 2x)(y - x)$

$= (x^2 - 4 - 2x)(x^2 - 4 - x)$ $(\because y = x^2 - 4)$

$= (x^2 - 2x - 4)(x^2 - x - 4)$

Type – IV: Factorization of the expression of the forms

- $a^3 + 3a^2b + 3ab^2 + b^3$
- $a^3 - 3a^2b + 3ab^2 - b^3$

Example 15: Factorize: $8x^3 + 60x^2 + 150x + 125$

Solution: $8x^3 + 60x^2 + 150x + 125$

$= (2x)^3 + 3(2x)^2(5) + 3(2x)(5)^2 + (5)^3$

$= (2x + 5)^3$

$= (2x + 5)(2x + 5)(2x + 5)$

Example 16: **Factorize: $x^3 - 18x^2 + 108x - 216$**

Solution: $x^3 - 18x^2 + 108x - 216$

$= (x)^3 - 3(x)^2(6) + 3(x)(6)^2 - (6)^3$

$= (x-6)^3$

$= (x-6)(x-6)(x-6)$

Type – V: Factorization of the expression of the form $a^3 \underline{+} b^3$

The expression $a^3 + b^3$ is a sum of cubes and it can be factorized using the following identity:

$$a^3 + b^3 = (a + b)(a^2 - ab + b^2)$$

The expression $a^3 - b^3$ is a difference of cubes and it can be factorized using the following identity:

$$a^3 - b^3 = (a - b)(a^2 + ab + b^2)$$

Example 17: **Factorize: $8x^3 + 27$**

Solution: $8x^3 + 27$

$= (2x)^3 + (3)^3$

$= (2x + 3)[(2x)^2 - (2x)(3) + (3)^2]$

$= (2x + 3)(4x^2 - 6x + 9)$

Example 18: **Factorize: $x^3 - 27y^3$**

Solution: $x^3 - 27y^3$

$= (x)^3 - (3y)^3$

$= (x - 3y)[(x)^2 + (x)(3y) + (3y)^2]$

$= (x - 3y)(x^2 + 3xy + 9y^2)$

EXERCISE 4.2

1. **Factorize each of the following expressions:**

(i) $4x^4 - 81y^4$ **(ii)** $a^4 - 64b^4$

(iii) $x^4 + 4x^2 + 4$ **(iv)** $x^4 - 14x^2 + 1$

(v) $x^4 - 30x^2y^2 + 9y^4$ **(vi)** $x^4 + 11x^2y^2 + y^4$

Solution: Let's factorize each expression:

(i) $4x^4 - 81y^4$

Sol: $4x + 81y^4$

$= (2x^2)^2 + (9y^2)^2 + 2(2x^2)(9y^2) - 2(2x^2)(9y^2)$

$= (2x^2 + 9y^2)^2 - 36x^2y^2$

$= (2x^2 + 9y^2)^2 - (6xy)^2$

$= (2x^2 + 9y^2 + 6xy)(2x^2 + 9y^2 - 6xy)$

$\boxed{\because a^2 - b^2 = (a+b)(a-b)}$

$= (2x^2 + 6xy + 9y^2)(2x^2 - 6xy + 9y^2)$

(ii) $a^4 - 64b^4$

Sol: $a^4 + 64b^4$

$= (a^2)^2 + (8b^2)^2 + 2(a^2)(8b^2) - 2(a^2)(8b^2)$

$= (a^2 + 8b^2)^2 - 16a^2b^2$

$= (a^2 + 8b^2)^2 - (4ab)^2$

$\boxed{\because a^2 - b^2 = (a+b)(a-b)}$

$= (a^2 + 8b^2 + 4ab)(a^2 + 8b^2 - 4ab)$

$= (a^2 + 4ab + 8b^2)(a^2 - 4ab + 8b^2)$

(iii) $x^4 + 4x^2 + 4$

Sol: $x^4 + 4x^2 + 16$

Rearrange

$= x^4 + 16 + 4x^2$

$= (x^2)^2 + (4)^2 + 4x^2$

By adding and subtracting $2(x^2)(4)$

$= (x^2)^2 + (4)^2 + 2(x^2)(4) - 2(x^2)(4) + 4x^2$

$= (x^2 + 4)^2 - 8x^2 + 4x^2$

$= (x^2 + 4)^2 - 4x^2$

$= (x^2 + 4)^2 - (2x)^2$

$= (x^2 + 4 + 2x)(x^2 + 4 - 2x)$

$= (x^2 + 2x + 4)(x^2 - 2x + 4)$

(iv) $\mathbf{x^4 - 14x^2 + 1}$

Sol: $x^4 - 14x^2 + 1$

Rearrange

$= x^4 + 1 - 14x^2$

$= (x^2)^2 + (1)^2 - 14x^2$

By adding and subtracting $2(x^2)(1)$

$= (x^2)^2 + (1)^2 + 2(x^2)(1) - 2(x^2)(1) - 14x^2$

$= (x^2 + 1)^2 + 2x^2 - 14x^2$

$= (x^2 +)^2 - 16x^2$

$= (x^2 + 1)^2 - (4x)^2$

$\boxed{\because a^2 - b^2 = (a+b)(a-b)}$

$= (x^2 + 1 + 4x)(x^2 + 1 - 4x)$

$= (x^2 + 4x + 1)(x^2 - 4x + 1)$

(v) $\mathbf{x^4 - 30x^2y^2 + 9y^4}$

Sol: $x^4 - 30x^2y^2 + 9y^4$

$= x^4 + 9y^4 - 30x^2y^2$

$= (x^2)^2 + (3y^2)^2 + 2(x^2)(3y^2) - 2(x^2)(3y^2) - 30x^2y^2$

$= (x^2 + 3y^2)^2 - 6x^2y^2 - 30x^2y^2$

$= (x^2 + 3y^2)^2 - 36x^2y^2$

$= (x^2 + 3y^2) - (6xy)^2$

$\boxed{\because a^2 - b^2 = (a+b)(a-b)}$

$= (x^2 + 3y^2 + 6xy)(x^2 + 3y^2 - 6xy)$

$= (x^2 + 6xy + 3y^2)(x^2 - 6xy + 3y^2)$

(vi) $\mathbf{x^4 - 11x^2 + y^4}$ **(Corrected)**

Sol: $x^4 - 11x^2y^2 + y^4$

$= x^4 + y^4 - 11x^2y^2$

$= (x^2)^2 + (y^2)^2 - 2(x^2)(y^2) + 2(x^2)(y^2) - 11x^2y^2$

$= (x^2 - y^2)^2 - 9x^2y^2$

$= (x^2 - y^2)^2 - (3xy)^2$

$= (x^2 - y^2 + 3xy)(x^2 - y^2 - 3xy)$

$= (x^2 + 3xy - y^2)(x^2 - 3xy - y^2)$

Factorize $(x+1)(x-1)(x+2)(x-2)+13x^2$

2. Factorize each of the following expressions:

(i) $(x + 1)(x + 2)(x + 3)(x + 4) + 1$

(ii) $(x + 2)(x - 7)(x - 4)(x - 1) + 17$

(iii) $(2x^2 + 7x + 3)(2x^2 + 7x + 5) + 1$

(iv) $(3x^2 + 5x + 3)(3x^2 + 5x + 5) - 3$

(v) $(x + 1)(x + 2)(x + 3)(x + 6) - 3x^2$

(vi) $(x + 1)(x - 1)(x + 2)(x - 2) + 5x^2$

(i) $\mathbf{(x + 1)(x + 2)(x + 3)(x + 4) + 1}$

Sol: Expression $= (x + 1)(x + 4)(x + 2)(x + 3) + 1$

$= (x^2 + 5x + 4)(x^2 + 5x + 6) + 1$

Put $x^2 + 5x = y$

Exp. $= (y + 4)(y + 6) + 1$

$= y^2 + 6y + 4y + 24 + 1$

$= y^2 + 10y + 25$

$= (y)^2 + 2(y)(5) + (5)^2$

$= (y + 5)^2$

$= (x^2 + 5x + 5)^2$

(ii) $\mathbf{(x + 2)(x - 7)(x - 4)(x - 1) + 17}$

$= (x^2 - 7x + 2x - 14)(x^2 - x - 4x + 4) + 17$

$= (x^2 - 5x - 14)(x^2 - 5x + 4) + 17$

Put $x^2 - 5x = y$

Exp. $= (y - 14)(y + 4) + 17$

$= y^2 + 4y - 14y - 56 + 17$

$= y^2 - 10y - 39 = (y^2 - 13y + 3y - 39)$

$= y(y - 13) + 3(y - 13) = (y + 3)(y - 13)$

$= (x^2 - 5x + 3)(x^2 - 5x - 13)$

(iii) $\mathbf{(2x^2+7x+3)(2x^2+7x+5)+1}$

Put $2x^2+7x=y$

Exp. $=(y+3)(y+5)+1$

$=y^2+8y+15+1$

$=y^2+8y+16$

$=(y+4)^2$

$=(2x^2+7x+4)(2x^2+7x+4)$

(iv) $\mathbf{(3x^2+5x+3)(3x^2+5x+5)-3}$

Put $3x^2+5x=y$

Exp. $=(y+3)(y+5)-3$

$=y^2+8y+15-3$

$=y^2+8y+12$

$=y^2+6y+2y+12$

$=y(y+6)+2(y+6)$

$=(y+2)(y+6)$

$=(3x^2+5x+2)(3x^2+5x+6)$

(v) $\mathbf{(x+1)(x+2)(x+3)(x+6)-3x^2}$

$=(x+1)(x+6)(x+2)(x+3)-3x^2$

$=(x^2+7x+6)(x^2+5x+6)-3x^2$

Put $=x^2+6=y$

$=(y+7x)(y+5x)-3x^2$

$=y^2+12xy+35x^2-3x^2$

$=y^2+12xy+32x^2$

$=y^2+8xy+4xy+32x^2$

$=y(y+8x)+4x(y+8x)$

$=(y+4x)(y+8x)$

$=(x^2+6+4x)(x^2+6+8x)$

$=(x^2+4x+6)(x^2+8x+6)$

(vi) $\mathbf{(x+1)(x-1)(x+2)(x-2)+5x^2}$

Re-arranging the terms

$(x+1)(x-2)(x+2)(x-2)+5x^2$

$=(x+1)(x+2)(x-1)(x-2)+5x^2$

$=(x^2+3x+2)(x^2-3x+2)+5x^2$

$=(y+3x)(y-3x)+5x^2$

$=y^2-9x^2+5x^2=y^2-4x^2$

$=(y+2x)(y-2x)$

$=(x^2+2x+2)(x^2-2x+2)$

3. Factorize:

(i) $\mathbf{8x^3+12x^2+6x+1}$

(ii) $\mathbf{27a^3+108a^2b+144ab^2+64b^3}$

(iii) $\mathbf{x^3+18x^2y+108xy^2+216y^3}$

(iv) $\mathbf{8x^3-125y^3+150xy^2-60x^2y}$

Sol: **(i)** $\mathbf{8x^3+12x^2+6x+1}$

$=(2x)^3+3(2x)^2(1)+3(2x)(1)^2+(1)^3$

$$\boxed{\because a^2+3a^2b+3ab^2+b^3=(a+b)^3}$$

$=(2x+1)^3$

(ii) $\mathbf{27a^3+108a^2b+144ab^2+64b^3}$

Sol: $27a^3+108a^2b+144ab^2+64b^3$

$=(3a)^3+3(3a)^2(4b)+3(3a)(4b)^2+(4b)^3$

$=(3a+4b)^3$

(iii) $\mathbf{x^3+48x^2y+108xy^2+216y^3}$ **(Corrected)**

Sol: $x^3+18x^2y+108xy^2+216y^3$

$=(x)^3+3(x)^2(6y)+3(x)(6y)^2+(6y)^3$

$=(x+6y)^3$

(iv) $\mathbf{8x^3-125y^3+150xy^2-60x^2y}$

Sol: $8x^3-125y^3+150xy^2-60x^2y$

$=8x^3-60x^2y+150xy^2-125y^3$

$=(2x)^3-3(2x)^2(5y)+3(2x)(5y)^2-(5y)^3$

$=(2x-5y)^3$

4. Factorize:

(i)	$\mathbf{125a^3-1}$	**(ii)**	$\mathbf{64x^3+125}$
(iii)	$\mathbf{x^6-27}$	**(iv)**	$\mathbf{1000a^3+1}$
(v)	$\mathbf{343x^3+216}$	**(vi)**	$\mathbf{27-512y^3}$

Solution: Let's factorize each expression step by step:

(i) $125a^3 - 1$

$= (5a)^3 - (1)^3$

$= (5a - 1)(25a^2 + 5a + 1)$

(ii) $64x^3 + 125$

$= (4x)^3 + (5)^3$

$= (4x + 5)(16x^2 - 20x + 25)$

(iii) $x^6 - 27$

$= (x^2)^3 - (3)^3 = (x^2 - 3)(x^4 + 3x^2 + 9)$

(iv) $1000a^3 + 1$

$(10a)^3 + (1)^3 = (10a + 1)(100a^2 - 10a + 1)$

(v) $343x^3 + 216$

$= (7x)^3 + (6)^3$

$= (7x + 6)(49x^2 - 42x + 36)$

(vi) $27 - 512y^3$

$= (3)^3 - (8y)^3$

$\Rightarrow (3 - 8y)(9 + 24y + 64y^2)$

Highest Common Factor (HCF)

The HCF of algebraic expressions refers to the greatest algebraic expression that divides two or more algebraic expressions without leaving a remainder.

We can find HCF of given expressions by the following two methods:

(a) By factorization (b) By division

(a) HCF by Factorization Method

Example 19: Find the HCF of $6x^2y$, $9xy^2$

Solution: $6x^2y = 2 \times 3 \times x \times x \times y$

$9xy^2 = 3 \times 3 \times x \times y \times y$

$\therefore$ HCF $= 3 \times x \times y$ (Product of common factors)

$= 3xy$

Example 20: Find the HCF by factorization method

$x^2 - 27, x^2 + 6x - 27, x^2 - 9$

Solution: $x^3 - 27 = x^3 - 3^3$

$= (x - 3)[(x)^2 + (3)(x) + (3)^2]$

$= (x - 3)(x^2 + 3x + 9)$

$x^2 + 6x - 27 = x^2 + 9x - 3x - 27$

$= x(x + 9) - 3(x + 9)$

$= (x + 9)(x - 3)$

$x^2 - 9 = x^2 - 3^2$

$= (x - 3)(x + 3)$

Hence, HCF $= x - 3$

(b) HCF by Division Method

Example 21: Find HCF of $6x^3 - 17x^2 - 5x + 6$ and $6x^3 - 5x^2 - 3x + 2$ by using division method.

Solution:

$$\begin{array}{r|l} & 1 \\ \hline 6x^3 - 17x^2 - 15x + 6 & 6x^3 - 5x^2 - 3x + 2 \\ & \underset{-}{}6x^3 \underset{+}{-} 17x^2 \underset{+}{-} 5x \underset{-}{+} 6 \\ \hline & 12x^2 + 2x - 4 \end{array}$$

Here, $12x^2 + 2x - 4 = 2(6x^2 + x - 2)$

2 is not common in both the given polynomials, so we ignore it and consider only $6x^2 + x - 2$.

$$\begin{array}{r|l} & x - 3 \\ \hline 6x^2 + x - 2 & 6x^2 - 17x^2 - 5x + 6 \\ & \underset{-}{}6x^3 \underset{-}{+} x^2 \underset{+}{-} 2x \\ \hline & -18x^2 - 3x + 6 \\ & \underset{+}{-}18x^2 \underset{+}{-} 3x \underset{-}{+} 6 \\ \hline & 0 \end{array}$$

Hence, HCF $= 6x^2 + x - 2$

Least Common Multiple (LCM)

The LCM of two or more algebraic expressions is the smallest expression that is divisible by each of the given expressions.

To find the LCM by factorization, we use the formula.

LCM = Common factors × Non-common factors

Example 22: Find the LCM of $4x^2y$, $8x^3y^2$.

Solution:

$4x^2y = 2 \times 2 \times x \times x \times y$

$8x^3y^2 = 2 \times 2 \times 2 \times x \times x \times x \times y \times y$

Common factors $= 2 \times 2 \times x \times x \times y = 4x^2y$

Non-common factors $= 2 \times x \times y = 2xy$

LCM = Common factors × Non-common factors

$= 4x^2y \times 2xy$

$= 8x^3y^2$

Example 23: Find the LCM of the polynomials

$x^2 - 3x + 2$, $x^2 - 1$ and $x^2 - 5x + 4$.

Solution: As $x^2 - 3x + 2 = x^2 - 2x - x + 2$

$= x(x-2) - 1(x-2)$

$= (x-2)(x-1)$

And $x^2 - 1 = (x-1)(x+1)$

$x^2 - 5x + 4 = x^2 - 4x - x + 4$

$= x(x-4) - 1(x-4)$

$= (x-4)(x-1)$

Common factors $= x - 1$

Non-common factors $= (x+1)(x-2)(x-4)$

LCM = Common factors × Non-common factors

$= (x-1) \times (x+1)(x-2)(x-4)$

$= (x-1)(x+1)(x-2)(x-4)$

Relationship between LCM and HCF

The relationship between LCM and HCF can be expressed as follows:

$$\text{LCM} \times \text{HCF} = p(x) \times q(x)$$

Where, $p(x) = 1^{st}$ polynomial

$q(x) = 2^{nd}$ polynomial

Example 24: LCM and HCF of two polynomials are $x^3 - 10x^2 + 11x + 70$ and $x - 7$. If one of the polynomials is $x^2 - 12x + 35$, find the other polynomial.

Solution: Given that: LCM $= x^3 - 10x^2 + 11x + 70$

HCF $= x - 7$

$p(x) = x^2 - 12x + 35$

As we know that: $q(x) = \dfrac{\text{LCM} \times \text{HCF}}{p(x)}$

$$= \frac{(x^3 - 10x^2 + 11x + 70)(x-7)}{x^2 - 12x + 35}$$

$$\begin{array}{r} x+2 \\ x^2 - 12x + 35\,\overline{)\,x^3 - 10x^2 + 11x + 70} \\ \underline{\underset{-}{x^3}\underset{+}{-}12x^2\underset{-}{+}35x} \\ 2x^2 - 24x + 70 \\ \underline{\underset{-}{2x^2}\underset{+}{-}24x\underset{-}{+}70} \\ 0 \end{array}$$

So, $q(x) = (x+2)(x-7)$

$= x^2 - 7x + 2x - 14$

$= x^2 - 5x - 14$

Example 25: The LCM of $x^2y + xy^2$ and $x^2 + xy$ is $xy(x+y)$. Find the HCF.

Solution: Given that: LCM $= xy(x+y)$

HCF = ?

1^{st} polynomial $= x^2y + xy^2$

2^{nd} polynomial $= x^2 + xy$

As we know that:

LCM × HCF = Product of two polynomials

$$\text{HCF} = \frac{\text{Product of two polynomials}}{\text{LCM}}$$

$$= \frac{(x^2y + xy^2)(x^2 + xy)}{xy(x+y)}$$

$$= \frac{xy(x+y)x(x+y)}{xy(x+y)}$$

$$= x(x+y)$$

EXERCISE 4.3

1. **Find HCF by factorization method.**

(i) $21x^2y, 35xy^2$ (ii) $4x^2-9y^2, 2x^2-3xy$

(iii) x^3-1, x^2+x+1

(iv) $a^3+2a^2-3a, 2a^3+5a^2-3a$

(v) $t^2-3t-4, t^2+5t+4, t^2-1$

(vi) $x^2+15x+56, x^2+5x-24, x^2+8x$

Solution:

(i) $21x^2y, 35xy^2$

- **Factorizing** $21x^2y$:

$$21x^2y = 3\times 7\times x\times x\times y$$

- **Factorizing** $35xy^2$:

$$35xy^2 = 5\times 7\times x\times y\times y$$

- **Finding the common factors:** The common factor is $7xy$.

HCF $= 7xy$

(ii) $4x^2-9y^2, 2x^2-3xy$

1. **Factorizing** $4x^2-9y^2$:

$$4x^2-9y^2 = (2x-3y)(2x+3y)$$

2. **Factorizing** $2x^2-3xy$:

$$2x^2-3xy = x(2x-3y)$$

3. **Finding the common factors:** The common factor is $(2x-3y)$.

HCF $= (2x-3y)$

(iii) x^3-1, x^2+x+1

1. **Factorizing** x^3-1:

$$x^3-1 = (x)^3-(1)^3 = (x-1)(x^2+x+1)$$

2. **Factorizing** x^2+x+1: This expression cannot be simplified further.

3. **Finding the common factors:** The common factor is (x^2+x+1).

HCF $= (x^2+x+1)$

(iv) **Given Expressions:**

$$a^3+2a^2-3a,\quad 2a^3+5a^2-3a$$

Step 1: Factorize each expression

FACTORIZING a^3+2a^2-3a:

$$a^3+2a^2-3a = a\cdot(a^2+2a-3)$$

Now factorize a^2+2a-3:

$$a^2+2a-3 = (a+3)(a-1)$$

Thus:

$$a^3+2a^2-3a = a\cdot(a+3)\cdot(a-1)$$

FACTORIZING $2a^3+5a^2-3a$:

$$2a^3+5a^2-3a = a\cdot(2a^2+5a-3)$$

Now factorize $2a^2+5a-3$:

$$2a^2+5a-3 = (a+3)(2a-1)$$

Thus:

$$2a^3+5a^2-3a = a\cdot(a+3)\cdot(2a-1)$$

Step 2: Identify common factors

The factorizations are:

$$a^3+2a^2-3a = a\cdot(a+3)\cdot(a-1)$$

$$2a^3+5a^2-3a = a\cdot(a+3)\cdot(2a-1)$$

The common factors are:

$$a \quad \text{and} \quad (a+3)$$

Step 3: Write the HCF

$$\text{HCF} = a(a+3)$$

(v) $t^2+3t+4, t^2+5t+4, t^2-1$

1. **Factors of** t^2+3t-4:

$= t^2+4t+t+4$

$= t(t+4)+1(t+4)$

$= (t+1)(t+4)$

2. **Factors of** t^2+5t+4:

$= t^2+4t+t+4$

$= t(t+4)+1(t+4)$

$= (t+1)(t+4)$

3. **Factors of** t^2-1:

$= (t)^2-(1)^2$

$= (t+1)(t-1)$

Common factors is $t+1$

$\therefore$ H.C.F $= t+1$

(vi) $\boldsymbol{x^2 + 15x + 56, x^2 + 5x - 24, x^2 + 8x}$

1. Factorizing $x^2 + 15x + 56$:

$x^2 + 15x + 56$
$= x^2 + 7x + 8x + 56$
$= x(x + 7) + 8(x + 7)$
$= (x + 7)(x + 8)$

2. Factorizing $x^2 + 5x - 24$:

$x^2 + 5x - 24$
$= x^2 + 8x - 3x - 24$
$= x(x + 8) - 3(x + 8)$
$= (x + 8)(x - 3)$

3. Factorizing $x^2 + 8x$:

$$x^2 + 8x = x(x + 8)$$

4. Finding the common factors: The common factor is $(x + 8)$.

HCF = $(x + 8)$

2. Find HCF of the following expressions by using division method:

(i) $27x^3 + 9x^2 - 3x - 9,\ 3x - 2$
(ii) $x^3 - 9x^2 + 21x - 15,\ x^2 - 4x + 3$
(iii) $2x^3 + 2x^2 + 2x + 2,\ 6x^3 + 12x^2 + 6x + 12$
(iv) $2x^3 - 4x^2 + 6x,\ x^3 - 2x,\ 3x^2 - 6x$

(i) $\boldsymbol{27x^3 + 9x^2 - 3x - 9, 3x - 2}$

Sol: H.CF by Division Method

$$\begin{array}{r}
9x^2 + 9x + 5 \\
3x - 2\,\overline{)\,\cancel{27x^3} + 9x^2 - 3x - 10} \\
\underline{-\cancel{27x^3} \mp 18x^2} \\
\cancel{27x^2} - 3x - 10 \\
\underline{-\cancel{27x^2} \mp 18x + 10} \\
\cancel{15x} - 10 \\
\underline{\cancel{\mp 15x} \mp 10} \\
0
\end{array}$$

HCF = $3x - 2$

(ii) $\boldsymbol{x^3 - 9x^2 + 23x - 15, x^2 - 4x + 3}$

H.C.F by Division Method

$$\begin{array}{r}
x - 5 \\
x^2 - 4x + 3\,\overline{)\,x^3 - 9x^2 + 23x - 15} \\
\underline{-x^3 \mp 4x^2 \pm 3x} \\
-5x^2 + 20x - 15 \\
\underline{\mp 5x^2 \pm 20x \mp 15} \\
0
\end{array}$$

H.C.F = $x^2 - 4x + 3$

(iii) $\boldsymbol{2x^3 + 2x^2 + 2x + 2, 6x^3 + 12x^2 + 6x + 12}$

H.C.F by Division Method

$$\begin{array}{r}
3 \\
2x^3 + 2x^2 + 2x + 2\,\overline{)\,6x^3 + 12x^2 + 6x + 12} \\
\underline{-6x^3 \pm 6x^2 \pm 6x \pm 6} \\
6x^2 + 6 \\
3\,(2x^2 + 2)
\end{array}
\quad
\begin{array}{r}
x + 1 \\
\overline{\big|\,2x^3 + 2x^2 + 2x + 2} \\
\underline{-2x^3 \pm 2x^2} \\
2x + 2 \\
\underline{-2x \pm 2} \\
\times
\end{array}$$

HCF = $2(x^2 + 1)$

(iv) $\boldsymbol{2x^3 - 4x^2 + 6x, x^3 - 2x, 3x^2 - 6x}$

First we find H.C.F of $x^3 - 2x,\ 3x^2 - 6x$

$$\begin{array}{r}
3 \\
x^2 - x\,\overline{)\,3x^2 - 6x} \\
\underline{\pm 3x^2 \mp 3x} \\
-3x
\end{array}
\qquad
\begin{array}{r}
x - 1 \\
x\,\overline{)\,x^2 - x} \\
\underline{\pm x^2} \\
-x \\
\underline{-x} \\
\times
\end{array}$$

HCF = ‘x’

3. Find LCM of the following expressions by using prime factorization method.

(i) $2a^2b,\ 4ab^2,\ 6ab$ (ii) $x^2 + x,\ x^3 + x^2$
(iii) $a^2 - 4a + 4,\ a^2 - 2a$ (iv) $x^4 - 16,\ x^3 - 4x$
(v) $16 - 4x^2,\ x^2 + x - 6,\ 4 - x^2$

Solution:

(i) $\mathbf{2a^2b, 4ab^2, 6ab}$

$\mathbf{2a^2b, 4ab^2, 6ab}$

$2a^2b = \textcircled{2}a \times \textcircled{a}\textcircled{b}$

$4ab^2 = \textcircled{2} \times 2 \times \textcircled{a}\textcircled{b} \times b$

$6ab = \textcircled{2} \times 3\ \textcircled{a}\textcircled{b}$

Product of common factors = 2ab

Product of non-common factors

$= (a)(2b)(3) = 6ab$

L.C.M $= (2ab)\ (6ab) = 12a^2b^2$

(ii) $\mathbf{x^2 + x, x^3 + x^2}$

$\mathbf{x^2 + x, x^3 + x^2}$

Factorization

$x^2 + x = \textcircled{x}\ (x + 1)$

$x^3 + x^2 = x^2\ (x + 1)$

$= \textcircled{x} \times x\ (x + 1)$

Product of common factors $= x(x + 1)$

Product of non-common factors $= x$

L.C.M $= x(x + 1) \times x$

L.C.M $= x^2(x + 1)$

(iii) $\mathbf{a^2 - 4a + 4, a^2 - 2a}$

Sol: $\mathbf{a^2 - 4a + 4, a^2 - 2a}$

Factorization

$a^2 - 4a + 4 = (a)^2 - 2(a)\ (2) + (2)^2$

$= (a - 2)^2$

$= (a - 2)\ (a - 2)$

$a^2 - 2a = a(a - 2)$

Product of common factors $= (a - 2)$

Product of non-common factors $= a(a - 2)$

L.C,M $= (a - 2)\ .\ a(a - 2)$

$= a(a - 2)^2$

(iv) $\mathbf{x^4 - 16,\ x^3 - 4x}$

Sol: $x^4 - 16,\ x^3 - 4x$

$x^4 - 16 = (x^2)^2 - (4)^2$

$= (x^2 + 4)\ (x^2 - 4)$

$= (x^2 + 4)\ [x^2 - 2^2]$

$= (x^2 + 4)\ (x + 2)\ (x - 2)$

Now, $x^3 - 4x = x(x^2 - 4) = x(x^2 - 2^2)$

$= x(x + 2)\ (x - 2)$

L.C.M $= (x + 2)\ (x - 2)\ x(x^2 + 4)$

$= (x^2 + 2)\ (x - 2)\ x(x^2 + 4)$

$= x(x^2 - 2^2)\ (x^2 + 4)$

$= x(x^2 - 4)\ (x^2 + 4)$

$= x(x^2)^2 - (4)^2$

$= x(x^4 - 16)$

(v) $\mathbf{16 - 4x^2, x^2 + x - 6, 4 - x^2}$ **(Corrected)**

Sol: $16 - 4x^2, x^2 + x - 6, x^2 - 4$

Factorization

$16 - 4x^2 = 4(4 - x^2)$

$= 4[2^2 - x^2]$

$= 4(2 + x)\ (2 - x)$

$= 4(x + 2)\ (-1)\ (-2 + x)$

$= (-1)\ (4)\ (x + 2)\ (x - 2)$

$= -4(x + 2)\ (x - 2)$

$x^2 + x - 6 = x^2 + 3x - 2x - 6$

$= x(x + 3) - 2\ (x + 3)$

$= (x + 3)\ (x - 2)$

$4 - x^2 = (2)^2 - (x)^2$

$= (2 + x)\ (2 - x)$

$= (x + 2)\ (-1)\ (-2 + x)$

$= -1(x + 2)\ (x - 2)$

Product of common factors

$(-1)\ (x + 2)\ (x - 2)$

$= (-1)\ (x^2 - 4) = (4 - x^2)$

Product of non-common factors $= 4(x + 3)$

L.C.M $= 4(x^2 - 4)\ (x + 3)$

4. The HCF of two polynomials is $y - 7$ and their LCM is $y^3 - 10y^2 + 11y + 70$. If one of the polynomials is $y^2 - 5y - 14$, find the other.

Solution: H.C.F $= y - 7$

L.C.M $= y^3 - 10y^2 + 11y + 70$

$p(y) = y^2 - 5y - 14$

We know that:

$$p(y)\,q(y) = \text{L.C.M} \times \text{H.C.F}$$

$$q(y) = \frac{\text{L.C.M} \times \text{H.C.F}}{p(y)}$$

$$q(y) = \frac{(y^3 - 10y^2 + 11y + 70) \times (y - 7)}{y^2 - 5y - 14}$$

$$q(y) = (y - 5)\,(y - 7)$$

$$q(y) = y^2 - 12y + 35$$

5. The LCM and HCF of two polynomial $p(x)$ and $q(x)$ are $36x^3(x + a)(x^3 - a^3)$ and $x^2(x - a)$ respectively. If $p(x) = 4x^2(x^2 - a^2)$, find $q(x)$.

Solution: L.C.M $= 36x^3\,(x + a)\,(x^3 - a^3)$

H.C.F $= x^2\,(x - a)$

$P(x) = 4x^2(x^2 - a^2)$

We know that

$p(x)\,.\,\text{q(x)} = \text{LCM} \times \text{H.C.F}$

$$q(x) = \frac{\text{L.C.M} \times \text{H.C.F}}{p(x)}$$

$$q(x) = \frac{36x^3(x + a)(x^3 - a^3) \times x^2(x - a)}{4x^2(x^2 - a^2)}$$

$$q(x) = \frac{9x^3(x^3 + a^3).x^2(x^2 - a^2)}{x^2(x^2 - a^2)}$$

$$q(x) = 9x^3(x^3 - a^3)$$

6. The HCF and LCM of two polynomials is $(x + a)$ and $12x^2(x + a)(x^2 - a^2)$ respectively. Find the product of the two polynomials.

Solution:

- The **HCF** of two polynomials is $(x + a)$.
- The **LCM** of the two polynomials is $12x^2(x + a)(x^2 - a^2)$.

Let $p(x)$ and $q(x)$ be required polynomials

We know that

$$p(x) \times q(x) = L.C.M \times H.C.F$$

$$p(x) \times q(x) = 12x^2(x + a)(x^2 - a^2) \times (x + a)$$

$$= 12x^2(x + a)^2(x + a) \times (x - a)$$

$$= 12x^2(x + a)^3(x - a)$$

Square Root of an Algebraic Expression

(a) Factorization Method

Example 26: Find the square root of the expression $36x^4 – 36x^2 + 9$

Solution:

$$36x^4 - 36x^2 + 9$$

$$= 9(4x^4 - 4x^2 + 1$$

$$= 9[(2x^2)^2 - 2(2x^2)(1) + (1)^2]$$

$$= 3^2(2x^2 - 1)^2$$

Taking square root on both sides

$$\sqrt{36x^4 - 36x^2 + 9} = \sqrt{3^2(2x^2 - 1)^2}$$

$$= \sqrt{3^2} \cdot \sqrt{(2x^2 - 1)^2}$$

$$= \pm 3(2x^2 - 1)$$

(b) Division Method

When the degree of the polynomial is higher, division method in finding the square root is very useful.

Example 27: Find the square root of the polynomial $x^4 - 12x^3 + 42x^2 - 36x + 9$.

Solution: Multiply x^2 by x^2 to get x^4

Multiply the quotient (x^2) by 2, so we get $2x^2$. By dividing $-12x^3$ by $2x^2$, we get $-6x$.

$$\begin{array}{r|l} & x^2 - 6x + 3 \\ \hline x^2 & x^4 - 12x^3 + 42x^2 - 36x + 9 \\ & \underline{-x^4} \\ 2x^2 - 6x & -12x^3 + 42x^2 \\ & \underline{\mp 12x^3 \pm 36x^2} \\ 2x^2 - 12x + 3 & 6x^2 - 36x + 9 \\ & \underline{-6x^2 \mp 36x \pm 9} \\ & 0 \end{array}$$

By continuing in this way, we get the remainder.
Hence, square root of $x^4 - 12x^3 + 42x^2 - 36x + 9$ is $\pm(x^2 - 6x + 3)$

Example 28: Cost function for producing a part is modeled by:

$$C(x) = 5x^2 - 25x + 30$$

Where x is the width of the component and $C(x)$ is the cost. Find the value of x where $C(x)$ is minimum.

Solution:
$$\begin{aligned} C(x) &= 5x^2 - 25x + 30 \\ &= 5(x^2 - 5x + 6) \\ &= 5(x^2 - 2x - 3x + 6) \\ &= 5[x(x-2) - 3(x-2)] \\ &= 5(x-2)(x-3) \end{aligned}$$

Thus, the minimum cost occurs when $x = 2$ and $x = 3$.

Example 29: The potential energy $U(x)$ of a particle moving in a cubic potential is represented as:

$$U(x) = x^3 - 6x^2 + 12x - 8$$

Factorize the expression to find the points where the energy in minimized.

Solution:
$$\begin{aligned} U(x) &= x^3 - 6x^2 + 12x - 8 \\ &= (x)^3 - 3(x)^2(2) + 3(x)(2)^2 - (2)^3 \\ &= (x-2)^3 \\ &= (x-2)(x-2)(x-2) \end{aligned}$$

The factorized form of the potential energy function shows that the energy is minimized at $x = 2$.

Example 30: A company's profit $p(x)$ is modeled by the quadratic equation:

$$P(x) = -5x^2 + 50x - 120$$

Where x represents the number of units produced and $P(x)$ represents the profit in dollars. Find how many units should be produced to maximize profit.

Solution:
$$\begin{aligned} P(x) &= -5x^2 + 50x - 120 \\ &= -5(x^2 - 10x + 24) \\ &= -5[x^2 - 4x - 6x + 24] \\ &= -5[x(x-4) - 6(x-4)] \\ &= -5(x-4)(x-6) \end{aligned}$$

We can see that profit will be 0 when $x = 4$ and $x = 6$. As coefficients of x^2 is negative, the maximum profit occurs at the midpoint between 4 and 6.

Which is: $$x = \frac{4+6}{2} = \frac{10}{2} = 5$$

Thus, the company should produce 5 units to maximize profit.

EXERCISE 4.4

1. **Find the square root of the following polynomials by factorization method:**

(i)	$x^2 - 8x + 16$	**(ii)**	$9x^2 + 12x + 4$
(iii)	$36a^2 + 84a + 49$	**(iv)**	$64y^2 - 32y + 4$
(v)	$200t^2 - 120t + 18$	**(vi)**	$40x^2 + 120x + 90$

Solution: **(i)** $x^2 - 8x + 16$

$$x^2 - 8x + 16 = (x^2) - 2(x)(4) + (4)^2 = (x-4)^2$$

Now, take the square root of both sides:

$$\sqrt{x^2 - 8x + 16} = \sqrt{(x-4)^2}$$
$$= \pm(x-4)$$

So, the square root is:

$$\pm(x-4)$$

(ii) $\mathbf{9x^2 + 12x + 4}$

$$9x^2 + 12x + 4 = 9$$
$$= (3x)^2 + 2(3x)(2) + (2)^2 = (3x+2)^2$$

Taking the square root of both sides:

$$\sqrt{9x^2 + 12x + 4} = \sqrt{(3x+2)^2}$$
$$= \pm(3x+2)$$

So, the square root is:

$$\pm(3x+2)$$

(iii) $\mathbf{36a^2 + 84a + 49}$

$$36a^2 + 84a + 49$$
$$= (6a)^2 + 2(6a)(7) + (7)^2$$
$$= (6a+7)^2$$

Now, take the square root of both sides:

$$\sqrt{36a^2 + 84a + 49} = \sqrt{(6a+7)^2}$$
$$= \pm(6a+7)$$

So, the square root is:

$$\pm(6a+7)$$

(iv) $\mathbf{64y^2 - 32y + 4}$

$$64y^2 - 32y + 4 = (8y)^2 - 2(8y)(2) + (2)^2 = (8y-2)^2$$

Now, take the square root of both sides:

$$\sqrt{64y^2 - 32y + 4} = \sqrt{(8y-2)^2}$$
$$= \pm(8y-2)$$

So, the square root is:

$$\pm(8y-2)$$

(v) $\mathbf{200t^2 - 120t + 18}$

$$200t^2 - 120t + 18 = 2(100t^2 - 60t + 9)$$
$$= 2[(10)^2 - 2(10t)(3) + (3)^2]$$
$$= 2(10t-3)^2$$

Now, take the square root of both sides:

$$\sqrt{200t^2 - 120t + 18} = \sqrt{2(10t-3)^2}$$
$$= \pm\sqrt{2}(10t-3)$$

So, the square root is:

$$\pm\sqrt{2}(10t-3)$$

(vi) $\mathbf{40x^2 + 120x + 90}$

$$40x^2 + 120x + 90 = 10(4x^2 + 12x + 9)$$
$$= 10[(2x)^2 + 2(2x)(3) + (3)^2]$$
$$= 10(2x+3)^2$$

Now, take the square root of both sides:

$$\sqrt{40x^2 + 120x + 90} = \sqrt{10(2x+3)^2}$$
$$= \pm\sqrt{10}\,(2x+3)$$

So, the square root is:

$$\pm\sqrt{10}\,(2x+3)$$

2. Find the square root of the following polynomials by division method:

(i) $\mathbf{4x^4 - 28x^3 + 37x^2 + 42x + 9}$

(ii) $\mathbf{121x^4 - 198x^3 - 183x^2 + 216x + 144}$

(iii) $\mathbf{x^4 - 10x^3y + 27x^2y^2 - 10xy^3 + y^4}$

(iv) $\mathbf{4x^4 - 12x^3 + 37x^2 - 42x + 49}$

Solution:

(i) $\mathbf{4x^4 - 28x^3 + 37x^2 + 42x + 9}$

Sol: Square root by division method:

$$\begin{array}{r|l}
 & 2x^2 - 7x - 3 \\ \hline
2x^2 & 4x^4 - 28x^3 - 37x^2 + 42x + 9 \\
 & \pm 4x^4 \\ \hline
4x^2 - 7x & -28x^3 + 37x^2 + 42x + 9 \\
 & \pm 28x^3 \pm 49x^2 \\ \cline{2-2}
4x^2 - 14x - 3 & -12x^2 + 42x + 9 \\
 & \mp 12x^2 \pm 42x \pm 9 \\ \cline{2-2}
 & 0
\end{array}$$

Thus, required square root is $\pm(2x^2 - 7x - 3)$

(ii) $\mathbf{121x^4 - 198x^3 - 183x^2 + 216x + 144}$

$$\begin{array}{r|l}
 & 11x^2 - 9x - 12 \\ \hline
11x^2 & 121x^4 - 198x^3 - 183x^2 + 216x + 144 \\
 & \pm 121x^4 \\ \hline
22x^2 - 9x & -198x^3 - 183x^2 + 216x + 144 \\
 & \mp 198x^3 \pm 81x^2 \\ \cline{2-2}
22x^2 - 18x - 12 & -264x^2 + 216x + 144 \\
 & \mp 264x^2 \pm 216x \pm 144 \\ \cline{2-2}
 & 0
\end{array}$$

Thus, required square root is $\pm(11x^2 - 9x - 12)$

(iii) $\mathbf{x^4 - 10x^3y + 27x^2y^2 - 10xy^3 + y^4}$

$$\begin{array}{r|l}
 & x^2 - 5xy + y^2 \\ \hline
x^2 & x^4 - 10x^3y + 27x^2y^2 - 10xy^3 + y^4 \\
 & \pm x^4 \\ \hline
2x^2 - 5xy & -10x^3y + 27x^2y^2 - 10xy^3 + y^4 \\
 & \mp 10x^3y \pm 25x^2y^2 \\ \cline{2-2}
2x^2 - 10xy + y^2 & 2x^2y^2 - 10xy^3 + y^4 \\
 & \pm 2x^2y^2 \mp 10xy^3 \pm y^4 \\ \cline{2-2}
 & 0
\end{array}$$

Thus, required square root is $\pm(x^2 - 5xy + y^2)$

(iv) $\mathbf{4x^4 - 12x^3 + 37x^2 - 42x + 49}$

$$\begin{array}{r|l}
 & 2x^2 - 3x + 7 \\ \hline
2x^2 & 4x^4 - 12x^3 + 37x^2 - 42x + 49 \\
 & \pm 4x^4 \\ \hline
4x^2 - 3x & -12x^3 + 37x^2 \\
 & \mp 12x^2 \pm 9x^2 \\ \cline{2-2}
4x^2 - 6x + 7 & 28x^2 - 42x + 49 \\
 & \pm 28x^2 \mp 42x \pm 49 \\ \cline{2-2}
 & 0
\end{array}$$

Thus, required square root is $\pm(2x^2 - 3x + 7)$

3. **An investor's return $R(x)$ in rupees after investing x thousand rupees is given by quadratic expression:**

$$R(x) = -x^2 + 6x - 8$$

Factorize the expression and find the investment levels that result in zero return.

Solution: $R(x) = -x^2 + 6x - 8$

$= -x^2 + 4x + 2x - 8$

$= -x(x-4) + 2(x-4)$

$= (-x+2)(x-4)$

$= (2-x)(x-4)$

Put $R(x) = 0$

$\Rightarrow \quad (2-x)(x-4) = 0$

$\Rightarrow \quad 2 - x = 0, x - 4 = 0$

$x = 2, x = 4$

4. **A company's profit $P(x)$ in rupees from selling x units of a product is modeled by the cubic expression:**

$$P(x) = x^3 - 15x^2 + 75x - 125$$

Find the break-even point(s), where the profit is zero.

Sol. $p(x) = x^3 - 15x^2 + 75x - 125$

Put $x = +5$ above

$p(-5) = (+5)3 - 15(+5)^2 + 75(+5) - 125$

$= +125 - 15 \times 25 + 375 - 125$

$= -375 + 375 = 0$

So $(x-5)$ is a factor of $p(x)$.

$$\begin{array}{r|l}
 & x^2 - 10x + 25 \\ \hline
x - 5 & x^3 - 15x^2 + 75y - 125 \\
 & \underset{-}{x^3} \mp 5x^2 \\ \cline{2-2}
 & -10x^2 + 75x - 125 \\
 & \underset{+}{\mp} 10x^2 \pm 50x \\ \cline{2-2}
 & 25x - 125 \\
 & \underset{-}{25x} \underset{+}{-} 125 \\ \cline{2-2}
 & x
\end{array}$$

$$= x^2 - 10x + 25$$
$$= (x - 5)^2$$

So

$$p(x) = x^3 - 15x^2 + 75x - 125$$
$$= (x - 5)(x - 5)(x - 5)$$

At $x = 5$

Profit is zero.

5. **The potential energy $V(x)$ in an electric field varies as a cubic function of distance x, given by:**

$$V(x) = 2x^3 - 6x^2 + 4x$$

Determine where the potential energy is zero.

Solution: $V(x) = 2x^3 - 6x^2 + 4x$

$$= 2x\,(x^2 - 3x + 2)$$
$$= 2x\,(x^2 - x - 2x + 2)$$
$$= 2x\,[x(x - 1) - 2(x - 1)]$$
$$= 2x\,(x - 1)(x - 2)$$

The factorized form of electric potential energy function shows that potential energy is zero at:

$$x = 0, x = 1, x = 2$$

6. **In structural engineering, the deflection $Y(x)$ of a beam is given by:**

$$y(x) = 2x^2 - 8x + 6$$

This equation gives the vertical deflection at any point x along the beam. Find the points of zero deflection.

Solution: $Y(x) = 2x^2 - 8x + 6$

$$= 2(x^2 - 4x + 3)$$
$$= 2(x^2 - 3x - x + 3)$$
$$= 2(x(x - 3) - 1(x - 3))$$
$$= 2(x - 1)(x - 3)$$

Points of zero deflection

$x - 1 = 0, x - 3 = 0$

$x = 1, \quad x = 3$

REVIEW EXERCISE 4

1. **Choose the correct option.**

i. The factorization of $12x + 36$ is:

(a) $12(x + 3)$ (b) $12(3x)$

(c) $12(3x + 1)$ (d) $x(12 + 36x)$

ii. The factors of $4x^2 - 12y + 9$ are:

(a) $(2x + 3)^2$ (b) $(2x - 3)^2$

(c) $(2x - 3)(2x + 3)$ (d) $(2 + 3x)(2 - 3x)^2$

iii. The HCF of a^3b^3 and ab^2 is:

(a) a^3b^3 (b) ab^2

(c) a^4b^5 (d) a^2b

iv. The LCM of $16x^2$, $4x$ and $30xy$ is:

(a) $480x^3y$ (b) $240xy$

(c) $240x^2y$ (d) $120x^4y$

v. Product of LCM and HCF = ________ of two polynomials.

(a) sum (b) difference

(c) product (d) quotient

vi. The square root of $x^2 - 6x + 9$ is:

(a) $\pm(x - 3)$ (b) $\pm(x + 3)$

(c) $x - 3$ (d) $x + 3$

vii. The LCM of $(a - b)^2$ and $(a - b)^4$ is:

(a) $(a - b)^2$ (b) $(a - b)^3$

(c) $(a - b)^4$ (d) $(a - b)^4$

viii. Factorization of $x^3 + 3x^2 + 3x + 1$ is:

(a) $(x + 1)^3$ (b) $(x - 1)^3$

(c) $(x+1)(x^2 + x + 1)$ (d) $(x - 1)(x^2 - x + 1)$

ix. Cubic polynomial has degree:

(a) 1 (b) 2

(c) 3 (d) 4

x. One of the factors of $x^3 - 27$ is:

(a) $x - 3$ (b) $x + 3$

(c) $x^2 - 3x + 9$ (d) Both *a* and *c*

Answers:

(i)	a	(ii)	b	(iii)	b	(iv)	c	(v)	c
(vi)	a	(vii)	c	(viii)	a	(ix)	c	(x)	a

2. Factorize the following expressions:

(i) $4x^3 + 18x^2 - 12x$ (ii) $x^3 + 64y^3$

(iii) $x^3y^3 - 8$ (iv) $-x^2 - 23x - 60$

(v) $2x^2 + 7x + 3$ (vi) $x^4 + 64$

(vii) $x^4 + 2x^2 + 9$

(viii) $(x + 3)(x + 4)(x + 5)(x + 6) - 360$

(ix) $(x^2 + 6x + 3)(x^2 + 6x - 9) + 36$

Solution: Let's factorize the given expressions:

(i) $\mathbf{4x^3 + 18x^2 - 12x}$

Sol: First, factor out the greatest common factor (GCF), which is $2x$:

$$4x^3 + 18x^2 - 12x = 2x(2x^2 + 9x - 6)$$

(ii) $\mathbf{x^3 + 64y^3}$

Sol: This is a sum of cubes, which can be factored using the formula $a^3 + b^3 = (a + b)(a^2 - ab + b^2)$. Here, $x^3 + (4y)^3$:

$$x^3 + 64y^3 = (x + 4y)(x^2 - 4xy + 16y^2)$$

(iii) $\mathbf{x^3y^3 - 8}$

Sol: This is also a difference of cubes, which can be factored using the formula $a^3 - b^3 = (a - b)(a^2 + ab + b^2)$. Here, $(xy)^3 - 2^3$:

$$x^3y^3 - 8 = (xy - 2)(x^2y^2 + 2xy + 4)$$

(iv) $\mathbf{-x^2 - 23x - 60}$

Sol: Factor out a negative sign:

$$-x^2 - 23x - 60 = -(x^2 + 23x + 60)$$

Now, factor $x^2 + 23x + 60$:

$$x^2 + 23x + 60 = (x + 3)(x + 20)$$

So the factorized form is:

$$-x^2 - 23x - 60 = -(x + 3)(x + 20)$$

(v) $\mathbf{2x^2 + 7x + 3}$

Sol: To factor, find two numbers that multiply to $2 \times 3 = 6$ and add to 7. These numbers are 6 and 1:

$$2x^2 + 7x + 3 = 2x^2 + 6x + x + 3$$

Now, factor by grouping:

$$2x(x + 3) + 1(x + 3) = (2x + 1)(x + 3)$$

(vi) $\mathbf{x^4 + 64}$

Sol: $x^4 + 64$

$$(x^2)^2 + (8)^2 + 2(x^2)(8) - 2(x^2)(8)$$

$$\boxed{\therefore\ (a+b) = a^2 + 2ab + b^2}$$

$$= (x^2 + 8)^2 - 16x^2$$

$$= (x^2 + 8)^2 - (4x^2)$$

$$\boxed{\therefore\ a^2 - b^2 = (a+b)(a-b)}$$

$$= (x^2 + 8 + 4x)(x^2 + 8 - 4x)$$

$$= (x^2 + 4x + 8)(x^2 - 4x + 8)$$

(vii): $\mathbf{x^4 + 2x^2 + 9}$

Sol:

$$\doteq (x^2)^2 + 3^2 + 2(x^2)(3) - 6x^2 + 2x^2$$

$$= (x^2 + 3)^2 - 4x^2$$

$$= (x^2 + 3)^2 - (2x)^2$$

$$= (x^2 + 3 + 2x)(x^2 + 3 - 2x)$$

$$= (x^2 + 2x + 3)(x^2 - 2x + 3)$$

(viii) $\mathbf{(x + 3)(x + 4)(x + 5)(x + 6) - 360}$

Sol:

$$= (x+3)(x+60)(x+4)(x+5) - 360$$

$$= (x^2 + 9x + 18)(x^2 + 9x + 20) - 360$$

Let $x^2 + 9x = y$

$$= (y + 18)(y + 20) - 360$$

$$= y^2 + 18y + 20y + 360 - 360$$

$$= y^2 + 38y$$

$$= y(y + 38)$$

$$= (x^2 + 9x)(x^2 + 9x + 38)$$

$$= x(x + 9)(x^2 + 9x + 38)$$

(ix): $(x^2 + 6x + 3)(x^2 + 6x - 9) + 36$

Sol: **Step 1: Expand the product of the two binomials.**

$$(x^2 + 6x + 3)(x^2 + 6x - 9)$$
$$= (x^2 + 6x)^2 + 3(x^2 + 6x) - 9(x^2 + 6x)$$

Put $x^2 + 6x = y$

$$\text{Exp.} = (y + 3)(y - 9) + 36$$
$$= y^2 - 6y - 27 + 36$$
$$= y^2 - 6y + 9$$
$$= (y - 3)^2$$

Replace y:

$$= (x^2 + 6x - 3)^2$$

3. Find LCM and HCF by prime factorization method:

- **(i)** $4x^3 + 12x^2,\ 8x^2 + 16x$
- **(ii)** $x^3 + 3x^2 - 4x,\ x^2 - x - 6$
- **(iii)** $x^2 + 8x + 16, x^2 - 16$
- **(iv)** $x^3 - 9x, x^2 - 4x + 3$

Solution:

(i) $4x^3 + 12x^2$ **and** $8x^2 + 16x$

Step 1: Factorize both expressions.

First Expression: $4x^3 + 12x^2$

Factor out the GCF:

$$4x^3 + 12x^2 = 4x^2(x + 3)$$

Second Expression: $8x^2 + 16x$

Factor out the GCF:

$$8x^2 + 16x = 8x(x + 2)$$

Step 2: Find the HCF and LCM.

- **HCF**: The common factors between the two factorizations are $4x$.

So, **HCF** = $4x$.

- **LCM**: To find the LCM, multiply all the distinct factors from both expressions:

LCM = $8x^2(x + 3)(x + 2)$.

So, **LCM** = $8x^2(x + 3)(x + 2)$.

(ii) $x^3 + 3x^2 - 4x$ **and** $x^2 - x - 6$

$$x^3 + 3x^2 - 4x = x(x^2 + 3x - 4)$$
$$= x(x^2 + 4x - x - 4)$$
$$= x(x + 4) - 1(x + 4)$$
$$= x(x - 1)(x + 4)$$

$$x^2 - x - 6 = x^2 - 3x + 2x - 6$$
$$= x(x - 3) + 2(x - 3)$$
$$= (x - 3)(x + 2)$$

HCF = 1

LCM = $x(x - 1)(x + 4)(x + 2)(x - 3)$
$= x(x - 1)(x - 3)(x + 2)(x + 4)$

(iii) $x^2 + 8x + 16$ **and** $x^2 - 16$

Step 1: Factorize both expressions.

First Expression: $x^2 + 8x + 16$

This is a perfect square trinomial:

$$x^2 + 8x + 16$$
$$= (x)^2 + 2(x)(4) + (4)^2$$
$$= (x + 4)^2$$
$$= (x + 4)(x + 4)$$

Second Expression: $x^2 - 16$

This is a difference of squares:

$$x^2 - 16$$
$$= (x)^2 - (4)^2$$
$$= (x + 4)(x - 4)$$

Step 2: Find the HCF and LCM.

- **HCF**: The common factor between the two expressions is $(x + 4)$.

So, **HCF** = $x + 4$.

- **LCM**: To find the LCM, multiply all the distinct factors from both expressions:

LCM = $(x + 4)^2(x - 4)$.

So, **LCM** = $(x + 4)^2(x - 4)$.

(iv) **$x^3 - 9x$ and $x^2 - x - 6$**

- **Factor out the GCF (Greatest Common Factor):**

$$x^3 - 9x = x(x^2 - 9)$$

- **Factor $x^2 - 9$ as a difference of squares:**

$$\begin{aligned} &x^2 - 9 \\ &= (x)^2 - (3)^2 \\ &= (x + 3)(x - 3) \end{aligned}$$

- **Final factorization:**

$$x^3 - 9x = x(x + 3)(x - 3)$$

Second Expression: $x^2 - 4x + 3$

- **Factorize the quadratic expression:**

$$\begin{aligned} &x^2 - 4x + 3 \\ &= x^2 - 3x - x + 3 \\ &= x(x - 3) - 1\ (x - 3) \\ &= (x - 3)(x - 1) \end{aligned}$$

Step 2: Find the HCF (Highest Common Factor)

From the factorized forms:

- $x^3 - 9x = x(x + 3)(x - 3)$
- $x^2 - 4x + 3 = (x - 3)(x - 1)$

The **common factor** is $(x - 3)$.

Thus, the **HCF** is:

$$\text{HCF} = x - 3$$

Step 3: Find the LCM (Least Common Multiple)

- Factors in $x^3 - 9x$: x, $(x + 3)$, and $(x - 3)$
- Factors in $x^2 - 4x + 3$: $(x - 3)$ and $(x - 1)$

The **LCM** will be:

$$\text{LCM} = x(x + 3)(x - 3)(x - 1)$$

Step 4: Find the HCF and LCM.

- **HCF**: The common factor between the two expressions is $(x - 3)$.

So, **HCF** $= x - 3$.

- **LCM**: To find the LCM, multiply all the distinct factors from both expressions:

LCM $= x(x + 3)(x - 3)(x + 2)$.

So, **LCM** $= x(x + 3)(x - 3)(x + 2)$.

$$LCM = x(x + 2)(x^2 - 9)$$

4. **Find square root by factorization and division method of the expression $16x^4 + 8x^2 + 1$.**

Solution:

1. **Factorization Method:**

We start with the given expression:

$$16x^4 + 8x^2 + 1$$

This expression is a quadratic in x^2, which suggests that we can factor it as a perfect square trinomial. Let's rewrite the expression in terms of $y = x^2$, so we have:

$$16y^2 + 8y + 1$$

Now, factor this quadratic:

$$\begin{aligned} &16y^2 + 8y + 1 \\ &= (4y)^2 + 2(4y)(1) + (1)^2 \\ &= (4y + 1)^2 \end{aligned}$$

Substitute $y = x^2$ back:

$$\pm(4x^2 + 1)$$

Thus, the square root of $16x^4 + 8x^2 + 1$ is:

$$\sqrt{16x^4 + 8x^2 + 1} = \pm(4x^2 + 2)$$

5. **Huria is analyzing the total cost of her loan, modeled by the expression $C(x) = x^2 - 8x + 15$, where x represents the number of years. What is the optimal repayment period for Huria's loan?**

Solution: $C(x) = x^2 - 8x + 15$

$$\begin{aligned} C(x) &= x^2 - 5x - 3x + 15 \\ C(x) &= x(x - 5) - 3(x - 5) \\ &= (x - 5)\ (x - 3) \end{aligned}$$

The factorized form of expression shows that cost of loan is zero at $x = 5$ and $x = 3$.

So, the optional repayment period is 3 years or 5 years.