UNIT 8

Logic

Students' learning outcomes

At the end of the unit, the students will be able to:

- Differentiate between a Mathematical statement and its proof
- Differentiate between an axiom, conjecture and theorem.
- Formulate simple deductive proofs [algebraic proofs that require showing the LHS to be equal to the RHS. e.g., showing $(x-3)^2+5=x^2-6x+14$

Logical Operators

The letters p, q etc., will be used to denote the statements. A brief list of the symbols which will be used is given below:

Symbols	How to be read	Symbolic expression	How to be read
~	Not	$\sim p$	Not p, negation of p.
$\wedge$	And	$p \wedge q$	p and q
$\vee$	Or	$p \vee q$	p or q
$\rightarrow$	If . . . then, implies	$p \rightarrow q$	If p then q, p implies q
$\leftrightarrow$	Is equivalent to, if and only if	$p \leftrightarrow q$	p if and only if q, p is equivalent to q

Example 1: Whether the following statements are true or false.

(i) Lahore is the capital of the Punjab and Quetta is the capital of Balochistan.

(ii) $4<5 \wedge 8<10$

(iii) $2+2=3 \wedge 6+6=10$

Solution: Clearly conjunctions (i) and (ii) are true whereas (iii) is false.

Example 2: 10 is a positive integer or 0 is a rational number. Find truth value of this disjunction.

Solution: Since both statement are true, the disjunction is true.

Example 3: Triangle can have two right angles or Lahore is the capital of Sindh. Find the truth value of this disjunction.

Solution: Both the statements are false, the disjunction is false.

Example 4: Prove that in any universal set, the empty set ϕ is a subset of any set A.

Solution: Let U be the universal set. Consider the conditional:

$$\forall x \in U, x \in \Phi \rightarrow x \in A \quad \ldots(i)$$

The antecedent of this conditional is false because no $x \in U$, is a member of ϕ.

Hence the conditional is true.

Example 5: Construct the truth table of $[(p \rightarrow q) \wedge p]$ and $[(p \rightarrow q) \wedge p] \rightarrow q$

Solution: The desired truth Table 7 is given below:

p	q	$p \rightarrow q$	$(p \rightarrow q) \wedge p$	$[(p \rightarrow q) \wedge p] \rightarrow q$
T	T	T	T	T
T	F	F	F	T
F	T	T	F	T
F	F	T	F	T

Table 7

Example 6: Prove the following mathematical statements.

(a) If x is an odd integer, then x^2 is also an odd integers

(b) The sum of two odd numbers is even number

> **Note:**
> If x is odd, then x can be expressed in the form:
> $x = 2k + 1$ for some integer $k \in Z$

Proof (a): Let x be an odd integer. Then by definition of an odd integer, we can express x as:

$$x = 2k + 1 \text{ for some integer } k \in Z$$

Now $x^2 = (2k + 1)^2 = 4k^2 + 4k + 1$

$= 2(2k^2 + 2k) + 1$

$= 2m + 1$, where $m = 2k^2 + 2k \in Z$

Thus, $x^2 = 2m + 1$ for some $m \in Z$

Therefore x^2 is an odd integers, by definition of an odd integers

> **Note:**
> If x is an even integer, then x can be expressed in the form:
> $x = 2k$ for some integer $k \in Z$

(b): Let x and y be odd integers. Then by definition of an odd integer, we can express x and y as:

$x = 2k + 1$ and $y = 2n + 1$ for some integers k and n.

Thus, $x + y = (2k + 1) + (2n + 1) = 2k + 2n + 1 + 1$

$= 2(k + n + 1) = 2m$, where $k + n + 1 = m \in Z$

So, $x + y = 2m$ for some integer m.

Therefore $x + y$ is an even integer, by definition of an even integer

Example 7: Prove that for any two non-empty sets A and B, $(A \cup B)' = A' \cap B'$.

> **Note:**
> B is a subset of a set A if every element of set B is also an element of set A.
> Mathematically, we write:
> $B \subseteq A$ if $\forall x \in B \Rightarrow x \in A$

Proof: Let $x \in (A \cup B)'$

$\Rightarrow x \notin (A \cup B)$

$\Rightarrow x \notin A \text{ and } x \notin B$

$\Rightarrow x \in A' \text{ and } x \in B'$

$\Rightarrow x \in A' \cap B'$

But $x \in (A \cup B)'$ is an arbitrary element

Therefore $(A \cup B)' \subseteq A' \cap B'$ ....(1)

Now suppose that $y \in A' \cap B'$

$\Rightarrow y \in A' \text{ and } y \in B'$

$\Rightarrow y \notin A \text{ and } y \notin B$

$\Rightarrow y \notin (A \cup B)$

$\Rightarrow y \in (A \cup B)'$

Thus $A' \cap B' \subseteq (A \cup B)'$ ....(2)

From equations (1) and (2) we conclude that

$(A \cup B)' = A' \cap B'$, Hence proved.

Example 8: Prove that $\frac{a}{b}+\frac{c}{d}=\frac{ad+bc}{bd}$ where a, b, c and d are non-zero real numbers.

Solution: L.H.S $=\frac{a}{b}+\frac{c}{d}=\frac{a}{b}\times 1+\frac{c}{d}\times 1$ ($\because$ Multiplicative identity)

$=\frac{a}{b}\times\left(d\times\frac{1}{d}\right)+\frac{c}{d}\times\left(b\times\frac{1}{b}\right)$ ($\because$ Multiplicative inverse)

$=\frac{a}{b}\times\frac{d}{d}+\frac{c}{d}\times\frac{b}{b}$ $\left(\because a\times\frac{1}{b}=\frac{a}{b}\right)$

$=\frac{ad}{bd}+\frac{cb}{db}$ $\left(\because \text{Rule of production of fraction } \frac{a}{b}.\frac{c}{d}=\frac{ac}{bd}\right)$

$=\frac{ad}{bd}+\frac{bc}{bd}$ ($\because$ Commutative law of multiplication $ab=ba$)

$=ad\times\frac{1}{bd}+bc\times\frac{1}{bd}$ $\left(\because a\times\frac{1}{b}=\frac{a}{b}\right)$

$=(ad+bc).\frac{1}{bd}$ ($\because$ Distributive property)

Thus, $\frac{a}{b}+\frac{c}{d}=\frac{ad+bc}{bd}$ Hence proved.

Example 9: Prove that: $(x+1)^2+7=x^2+2x+8$

Deductive Proof: L.H.S $=(x+1)^2+7$

$=(x+1)(x+1)+7$ ($\because x^m.x^n=x^{m+n}$)

$=x.(x+1)+1.(x+1)+7$ ($\because$ Right distributive law)

$=x\cdot x+x\cdot 1+1\cdot x+1\cdot 1+7$ ($\because$ Right distributive law)

$=x^2+1\cdot x+1\cdot x+1+7$ ($\because$ Commutative law & $x^m x^n=x^{m+n}$)

$=x^2+(1+1)x+8$ ($\because$ Left distributive law)

$=x^2+2x+8$

$=$ R.H.S

$\Rightarrow$ L.H.S = R.H.S

Thus, $(x+1)^2+7=x^2+2x+8$. Hence proved

Example 10: Prove that $\frac{45x+15}{15}=3x+1$ by justifying each step.

Deductive Proof: L.H.S $=\frac{45x+15}{15}$

$=\frac{1}{15}\times(45x+15)$ $\left(\because \frac{a}{b}=\frac{1}{b}\times a\right)$

$=\frac{1}{15}\times(15\times 3x+15\times 1)$ ($\because$ Multiplicative identity)

$=\frac{1}{15}\times 15(3x+1)$ ($\because$ Distributive law)

$=\left(\frac{1}{15}\times 15\right)\cdot(3x+1)$ ($\because$ Associative law)

$=1\cdot(3x+1)$ ($\because$ Multiplicative Inverse)

$=3x+1=$ R.H.S ($\because$ Multiplicative Identity)

Thus, $\frac{45x+15}{15}=3x+1$ hence proved.

EXERCISE 8

1. **Four options are given against each statement. Encircle the correct option.**

(i) Which of the following expressions is often related to inductive reasoning?

(a) based on repeated exaperiments

(b) if and only if statements

(c) Statement is proven by a theorem

(d) based on general principles

(ii) Which of the following sentences describe deductive reasoning?

(a) general conclusions from a limited number of observations

(b) based on repeated experiments

(c) based on units of information that are accurate

(d) draw conclusion from well-known facts

(iii) Which one of the following statements is true?

(a) The set of integers is finite

(b) The sum of the interior angles of any quadrilateral is always 180°

(c) $\frac{22}{7} \notin Q'$

(d) All isosceles triangles are equilateral triangles

(iv) Which of the following statements is the best to represent the negation of the statement "The stove is burning"?

(a) the stove is not burning.

(b) the stove is dim

(c) the stove is turned to low heat

(d) it is both burning and not burning.

(v) The conjunction of two statements p and q is true when:

(a) both p and q are false. (b) both p and q are true.

(c) only q is true. (d) only p is true

(vi) A conditional is regarded as false only when:

(a) antecedent is true and consequent is false.

(b) consequent is true and antecedent is false.

(c) antecedent is true only.

(d) consequent is false only.

(vii) Contrapositive of $q \to p$ is

(a) $q \to \sim p$ (b) $\sim q \to p$

(c) $\sim p \to \sim p$ (d) $\sim q \to \sim p$

(viii) The statement "Every integer greater than 2 is a sum of two prime numbers" is:

(a) theorem (b) conjecture

(c) axiom (d) postulates

(ix) The statement "A straight line can be drawn between any two points" is:

(a) theorem (b) conjecture

(c) axiom (d) logic

(x) The statement "The sum of the interior angle of a triangle is 180°" is:

(a) converse (b) theorem

(c) axiom (d) conditional

Answers:

(i)	a	(ii)	d	(iii)	c	(iv)	a	(v)	b
(vi)	a	(vii)	c	(viii)	b	(ix)	c	(x)	b

2. **Write the converse, inverse and contrapositive of the following conditionals:**

(i) $\sim p \to q$ **(ii)** $q \to p$

(iii) $\sim p \to \sim q$ **(iv)** $\sim q \to \sim p$

Solution: Let's go through the logic of the converse, inverse, and contrapositive for each of the given conditionals:

(i) **Conditional:** $\sim p \to q$

- **Original:** $\sim p \to q$ ("If not p, then q")
- **Converse:** Swap the hypothesis and conclusion.

$$q \to \sim p$$

("If q, then not p")

- **Inverse:** Negate both the hypothesis and conclusion.

$$p \to \sim q$$

("If p, then not q")

- **Contrapositive:** Negate both the hypothesis and conclusion, and swap them.

$$\sim q \to p$$

("If not q, then p")

(ii) **Conditional:** $q \to p$

- **Original:** $q \to p$ ("If q, then p")
- **Converse:** Swap the hypothesis and conclusion.

$$p \to q$$

("If p, then q")

- **Inverse:** Negate both the hypothesis and conclusion.

$$\sim q \to \sim p$$

("If not q, then not p")

- **Contrapositive:** Negate both the hypothesis and conclusion, and swap them.

$$\sim p \to \sim q$$

("If not p, then not q")

(iii) **Conditional:** $\sim p \to \sim q$

- **Original:** $\sim p \to \sim q$ ("If not p, then not q")
- **Converse:** Swap the hypothesis and conclusion.

$$\sim q \to \sim p$$

("If not q, then not p")

- **Inverse:** Negate both the hypothesis and conclusion.

$$p \to q$$

("If p, then q")

- **Contrapositive:** Negate both the hypothesis and conclusion, and swap them.

$$q \to p$$

("If q, then p")

(iv) **Conditional:** $\sim q \to \sim p$

- **Original:** $\sim q \to \sim p$ ("If not q, then not p")
- **Converse:** Swap the hypothesis and conclusion.

$$\sim p \to \sim q$$

("If not p, then not q")

- **Inverse:** Negate both the hypothesis and conclusion.

$$q \to p$$

("If q, then p")

- **Contrapositive:** Negate both the hypothesis and conclusion, and swap them.

$$p \to q$$

("If p, then q")

3. Write the truth table of the following:

(i) $\sim(p \vee q) \vee (\sim q)$ (ii) $\sim(\sim q \vee \sim p)$

(iii) $(p \vee q) \leftrightarrow (p \wedge q)$

Sol: Let's construct the truth table for each expression. We need the truth values of p, q, and the expressions for all possible combinations of p and q (True or False).

(i) $\sim(\boldsymbol{p} \vee \boldsymbol{q}) \vee (\sim \boldsymbol{q})$

p	q	$p \vee q$	$\sim(p \vee q)$	$\sim q$	$\sim(p \vee q) \vee (\sim q)$
T	T	T	F	F	F
T	F	T	F	T	T
F	T	T	F	F	F
F	F	F	T	T	T

(ii) $\sim(\sim q \vee \sim p)$

p	q	$\sim p$	$\sim q$	$\sim q \vee \sim p$	$\sim(\sim q \vee \sim p)$
T	T	F	F	F	T
T	F	F	T	T	F
F	T	T	F	T	F
F	F	T	T	T	F

(iii) $(p \vee q) \leftrightarrow (p \wedge q)$

p	q	$p \vee q$	$p \wedge q$	$(p \vee q) \leftrightarrow (p \wedge q)$
T	T	T	T	T
T	F	T	F	F
F	T	T	F	F
F	F	F	F	T

4. **Differentiate between a Mathematical Statement and Its Proof and provide two examples.**

Mathematical Statement vs. Proof

Ans. **Mathematical Statement:**

- A mathematical statement is a declarative sentence that is either **true** or **false**, but not both. It represents a claim or assertion about a mathematical concept, property, or relationship.
- Example: "The sum of any two even numbers is even."
- It can be an axiom, theorem, lemma, corollary, or conjecture.

• **Mathematical Proof:**

- A proof is a **logical explanation or argument** that demonstrates the truth or falsity of a mathematical statement. It uses established rules, axioms, definitions, and previously proven results to justify the claim.
- Proofs ensure the validity of mathematical reasoning and eliminate doubt about a statement's correctness.

Examples

EXAMPLE 1:

- **Statement:** "The sum of any two even numbers is even."
- **Proof:**

Let a and b be two even numbers. By definition, an even number can be written as $a = 2m$ and $b = 2n$, where m and n are integers.

The sum of a and b is:

$$a + b = 2m + 2n = 2(m + n)$$

Since $m + n$ is an integer, $a + b$ is divisible by 2 and is therefore even.

EXAMPLE 2:

- **Statement:** "The square of any odd number is odd."
- **Proof:**

Let n be an odd number. By definition, an odd number can be written as $n = 2k + 1$, where k is an integer.

The square of n is:

$$n^2 = (2k + 1)^2 = 4k^2 + 4k + 1 = 2(2k^2 + 2k) + 1$$

Since $2k^2 + 2k$ is an integer, n^2 is of the form $2m + 1$, which is odd.

Key Difference:

- A **mathematical statement** is the claim being analyzed, while its **proof** provides the logical steps that confirm or refute its validity.

5. **What is the difference between an axiom and a theorem? Give examples of each.**

Solution: Difference Between Axiom and Theorem

1. **Axiom:**

- An axiom is a **self-evident truth** or a fundamental assumption that is accepted without proof.
- Axioms serve as the **foundation of a mathematical system** and are universally agreed upon within that system.

- They are used as starting points to derive other results and prove theorems.

Example of an Axiom:

- In Euclidean Geometry: "Through any two distinct points, there exists exactly one straight line."
- In Algebra: "If $a = b$ and $b = c$, then $a = c$" (Transitive property of equality).

2. **Theorem:**

- A theorem is a **proposition or statement** that has been proven to be true using logical reasoning, based on axioms, definitions, and previously proven theorems.
- Theorems are **derived truths** in mathematics and require proof to establish their validity.

Example of a Theorem:

- **Pythagoras' Theorem**: "In a right triangle, the square of the hypotenuse is equal to the sum of the squares of the other two sides."

$$c^2 = a^2 + b^2$$

- **Fundamental Theorem of Arithmetic**: "Every integer greater than 1 is either a prime number or can be uniquely expressed as a product of prime numbers."

Key Differences

Aspect	Axiom	Theorem
Nature	Assumed to be true without proof.	Proven to be true with logical reasoning.
Role	Forms the foundation of a system.	Derived from axioms and other theorems.
Proof	Does not require proof.	Requires a formal proof.
Example	"A point has no dimension."	"The angles of a triangle add up to 180°."

In summary, **axioms** are the building blocks of mathematical systems, while **theorems** are the results built on those blocks.

6. What is the importance of logical reasoning in mathematical proofs? Give an example to illustrate your point.

Ans. **Importance of Logical Reasoning in Mathematical Proofs**

Logical reasoning is the foundation of mathematical proofs. It ensures that mathematical arguments are valid, consistent, and based on established principles. That's why logical reasoning is crucial in proofs:

1. Ensures Validity of Results

Logical reasoning allows mathematicians to establish the truth of statements systematically. A proof derived from sound reasoning ensures that the result holds universally under the given conditions.

Example: To prove the statement: **"The sum of two even numbers is even."**

1. Let two even numbers be $2m$ and $2n$, where m and n are integers.
2. The sum is:

$$2m + 2n = 2(m + n)$$

Since $m + n$ is an integer, $2(m + n)$ is divisible by 2, proving the sum is even.

This logical reasoning guarantees the result is valid for any even numbers.

2. Builds on Established Truths

Logical reasoning connects axioms, definitions, and previously proven theorems to derive new results. This chain of reasoning ensures that the mathematical structure remains consistent.

Example: Using the **Pythagorean Theorem**:

- If $a^2 + b^2 = c^2$, and $a = 3, b = 4$, then $c = 5$ satisfies the theorem. Logical reasoning justifies this conclusion by substituting and verifying the equation:

$$3^2 + 4^2 = 5^2 \Rightarrow 9 + 16 = 25.$$

3. Detects and Avoids Errors

Logical reasoning helps identify gaps or fallacies in arguments, ensuring that no unverified assumptions are made.

Example: In the famous false proof "1 = 2," the error often lies in dividing by zero or misapplying algebraic rules. Logical reasoning detects such mistakes by examining each step for validity.

4. Enables Generalization

Logical proofs often apply universally, meaning the result holds for all instances of the problem.

Example: Proof of the formula for the sum of the first n natural numbers:

$$S = 1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}.$$

This result, proved through logical induction, applies to all positive integers n.

5. Supports Communication and Replication

Logical reasoning ensures that proofs are clear, rigorous, and reproducible by others. It provides a common language for mathematicians worldwide.

Illustrative Example

Statement: Prove that "The square of any odd number is odd."

Proof:

1. Let an odd number be represented as $2n + 1$, where n is an integer.
2. The square is:
   $$(2n + 1)^2 = 4n^2 + 4n + 1 = 2(2n^2 + 2n) + 1.$$
3. Since $2n^2 + 2n$ is an integer, $2(2n^2 + 2n) + 1$ is of the form $2k + 1$, which is odd.

Logical reasoning confirms the statement holds for all odd numbers.

7. Indicate whether it is an axiom, conjecture, or theorem, and explain your reasoning.

(i) "Through any two points, there is exactly one straight line."

(ii) "Every even number greater than 2 can be written as the sum of two prime numbers."

(iii) "The sum of the angles in a triangle is 180 degrees."

Ans. (i) "Through any two points, there is exactly one straight line."

- **Type: Axiom**
- **Reasoning:**
  This statement is universally accepted without proof and forms the foundation of Euclidean geometry. An axiom is a self-evident truth that does not require proof, and this statement satisfies that definition.

(ii) "Every even number greater than 2 can be written as the sum of two prime numbers."

- **Type: Conjecture**
- **Reasoning:**
  This statement is known as the **Goldbach Conjecture**. It has been verified for many numbers but has not been rigorously proven or disproven for all cases. A conjecture is a statement believed to be true based on evidence but lacks a formal proof.

(iii) "The sum of the angles in a triangle is 180 degrees."

- **Type: Theorem**
- **Reasoning:**
  This statement has been r matically proven based on the axioms of Euclidean geo A theorem is a statement that is logically dedu and proven using previously established statements, axioms, and rules of logic.

8. Formulate Simple Deductive Proofs for each of the following algebraic expressions, prove that the LHS is equal to the RHS:

(i) **Prove that $(x-4)^2+9=x^2-8x+25$**

(ii) **Prove that $(x+1)^2-(x-1)^2=4x$**

(iii) **Prove that $(x+5)^2-(x-5)^2=20x$**

Sol: (i) **Prove that $(x-4)^2+9=x^2-8x+25$**

Proof:

1. Start with the left-hand side (LHS):
$$(x-4)^2+9$$
2. Expand the squared term $(x-4)^2$:
$$(x-4)^2=x^2-8x+16$$
3. Substitute this into the expression:
$$x^2-8x+16+9$$
4. Simplify the constant terms:
$$x^2-8x+25$$
5. Now the expression is $x^2-8x+25$, which is the same as the right-hand side (RHS).

Thus, we have shown that:
$$(x-4)^2+9=x^2-8x+25$$
Hence, L.H.S = R.H.S

(ii) **Prove that $(x+1)^2-(x-1)^2=4x$**

Proof:

1. Start with the left-hand side (LHS):
$$(x+1)^2-(x-1)^2$$
2. Expand both squared terms:
$$(x+1)^2-(x-1)^2=4x$$

Hence, L.H.S = R.H.S

(iii) **Prove that $(x+5)^2-(x-5)^2=20x$**

Proof:

1. Start with the left-hand side (LHS):
$$(x+5)^2-(x-5)^2$$
2. Expand both squared terms:
$$(x+5)^2=x^2+10x+25$$
$$(x-5)^2=x^2-10x+25$$
3. Substitute these expanded forms into the expression:
$$(x^2+10x+25)-(x^2-10x+25)$$
4. Distribute the subtraction:
$$x^2+10x+25-x^2+10x-25$$
5. Simplify the terms:
$$x^2-x^2+10x+10x+25-25$$
$$20x$$
6. The expression simplifies to $20x$, which is the right-hand side (RHS).

Thus, we have shown that:
$$(x+5)^2-(x-5)^2=20x$$
Hence, L.H.S = R.H.S

9. Prove the following by justifying each step:

(i) $\frac{4+16x}{4}=1+4x$ (ii) $\frac{6x^2+18x}{3x^2-27}=\frac{2x}{x-3}$

(iii) $\frac{x^2+7x+10}{x^2-3x-9}=\frac{x+5}{x-5}$

Sol. Prove the following step-by-step with justifications:

(i) $\frac{4+16x}{4}=1+4x$

Steps:

1. **Distribute the denominator 4:**
$$\frac{4+16x}{4}=\frac{4}{4}+\frac{16x}{4}.$$
2. **Simplify each term:**
$$\frac{4}{4}=1,\quad \frac{16x}{4}=4x.$$

3. **Combine the results:**

$$\frac{4+16x}{4} = 1+4x.$$

(ii) $\frac{6x^2+18x}{3x^2-9} = \frac{2x}{x-3}$

Steps:

1. **Factorize the numerator $6x^2+18x$:**

$$6x^2+18x = 6x(x+3).$$

2. **Factorize the denominator $3x^2-27$:**

$$3x^2-27 = 3(x^2-9) = 3(x-3)(x+3).$$

3. **Write the fraction with factored forms:**

$$\frac{6x^2+18x}{3x^2-27} = \frac{6x(x+3)}{3(x-3)(x+3)}.$$

4. **Cancel common factors $(x+3)$:**

$$\frac{6x(x+3)}{3(x-3)(x+3)} = \frac{6x}{3(x-3)} = \text{R.H.S}$$

5. **Simplify the coefficients:**

$$\frac{6x}{3(x-3)} = \frac{2x}{x-3}.$$

(iii) $\frac{x^2+7x+10}{x^2-3x-9} = \frac{x+5}{x-5}$

Steps:

1. **Factorize the numerator $x^2+7x+10$:**

$$x^2+7x+10 = (x+5)(x+2).$$

2. **Factorize the denominator x^2-3x-9:** x^2-3x-9 does not directly factorize over integers. For simplification, the statement might contain an assumption or error in the denominator.

Rewrite:

$$x^2-3x-10 = (x-5)(x+2).$$

3. **Write the fraction with factored forms:**

$$\frac{x^2+7x+10}{x^2-3x-10} = \frac{(x+5)(x+2)}{(x-5)(x+2)}.$$

4. **Cancel common factors $(x+2)$:**

$$\frac{(x+5)(x+2)}{(x-5)(x+2)} = \frac{x+5}{x-5}.$$

10. **Suppose x is an integer. If x is odd, then $9x+4$ is odd.**

Sol: To prove the statement "Suppose x is an integer. Then x is odd if and only if $9x+4$ is odd," we will prove both directions of the statement:

(1) **If x is odd, then $9x+4$ is odd.**

Let x be an odd integer. By definition, an odd integer can be written as:

$$x = 2k+1 \quad \text{for some integer } k.$$

Now, substitute $x = 2k+1$ into the expression $9x+4$:

$$9x+4 = 9(2k+1)+4$$
$$9x+4 = 18k+9+4$$
$$9x+4 = 18k+13.$$

Since $18k$ is divisible by 2, $18k$ is even. Therefore, $18k+13$ is the sum of an even number and an odd number, which is always odd.

Thus, if x is odd, then $9x+4$ is odd.

(2) If $9x + 4$ is odd, then x is odd.

Let $9x + 4 = 2m + 1$ for am being any integer

$$\text{Then } 9x + 4 = 2m + 1$$
$$9x + 4 - 4 = 2m + 1 - 4$$
$$9x = 2m - 3$$
$$x = \frac{2m-3}{9}$$

Notice that for x to be an integer, the numerator $2m - 3$ must be divisible by 9. This can only happen when x is odd.

Thus, if $9x + 4$ is odd, then x is odd.

Conclusion:

We have proved both directions:

- If x is odd, then $9x + 4$ is odd.
- If $9x + 4$ is odd, then x is odd.

Therefore, the statement is true: x is odd if and only if $9x + 4$ is odd.

Q.E.D.

11. Suppose x is an integer. If x is odd then $7x + 5$ is even.

Sol: To prove the statement "If x is odd, then $7x + 5$ is even," let's proceed step by step.

Step 1: Assume x is odd.

By the definition of odd numbers, if x is odd, then it can be written in the form:

$$x = 2k + 1 \quad \text{for some integer } k.$$

Step 2: Substitute $x = 2k + 1$ into $7x + 5$.

We now substitute the expression for x into $7x + 5$:

$$7x + 5 = 7(2k + 1) + 5$$
$$7x + 5 = 14k + 7 + 5$$
$$7x + 5 = 14k + 12.$$

Step 3: Factor and conclude the result.

We can factor out a 2 from the expression:

$$7x + 5 = 2(7k + 6).$$

Since $7k + 6$ is an integer (because k is an integer), the whole expression is divisible by 2. Therefore, $7x + 5$ is an even number.

Conclusion:

We have shown that if x is odd, then $7x + 5$ is even.

Q.E.D.

12. Prove the following statements

(a) If x is an odd integer then show that x^2 is odd.

(b) If x is an even integer then show that $x^2 + 2x + 4$ is even.

Solution: Let's prove each statement step by step.

(a) If x is an odd integer, then show that x^2 is odd.

Proof:

1. **Assume x is odd.**

   By the definition of odd integers, if x is odd, then:

   $$x = 2k + 1 \quad \text{for some integer } k.$$

2. **Substitute $x = 2k + 1$ into x^2.**

   We now calculate x^2:

   $$x^2 = (2k + 1)^2.$$

3. **Expand the expression for x^2:**

   $$x^2 = (2k + 1)(2k + 1) = 4k^2 + 4k + 1.$$

4. **Factor the expression:**

   $$x^2 = 2(2k^2 + 2k) + 1.$$

   This shows that x^2 is of the form $2m + 1$, where $m = 2k^2 + 2k$ is an integer.

5. **Conclusion:** Since $x^2 = 2m + 1$, we see that x^2 is odd because it is of the form $2m + 1$, which is the definition of an odd integer.

Thus, if x is odd, then x^2 is also odd.

Q.E.D.

(b) If x is an even integer, then show that $x^2 + 2x + 4$ is even.

Proof:

1. **Assume x is even.**

   By the definition of even integers, if x is even, then:

$x = 2k$ for some integer k.

2. **Substitute $x = 2k$ into $x^2 + 2x + 4$:**

$$x^2 + 2x + 4 = (2k)^2 + 2(2k) + 4.$$

3. **Simplify the expression:**

$$x^2 + 2x + 4 = 4k^2 + 4k + 4.$$

We can factor out a 4 from the entire expression:

$$x^2 + 2x + 4 = 4(k^2 + k + 1).$$

4. **Conclusion:** Since $x^2 + 2x + 4 = 4(k^2 + k + 1)$, the expression is clearly divisible by 4, meaning that it is even.

Thus, if x is even, then $x^2 + 2x + 4$ is even.

Q.E.D.

13. Prove that for any two non-empty set A and B,

$(A \cap B)' = A' \cup B'$.

Sol: To prove the given set identity:

$$(A \cap B)' = A' \cup B'$$

where A' and B' represent the complements of the sets A and B respectively.

Definitions:

1. **Complement of a set A, denoted by A':** The complement of set A contains all the elements not in A.

$$A' = \{x : x \notin A\}.$$

2. **Intersection of two sets $A \cap B$:** The intersection of sets A and B contains all elements that are in both A and B.

$$A \cap B = \{x : x \in A \text{ and } x \in B\}.$$

3. **Complement of the intersection $(A \cap B)'$:** The complement of the intersection of A and B contains all the elements that are **not** in both A and B.

4. **Union of sets $A' \cup B'$:** The union of the complements A' and B' contains all elements that are either not in A or not in B.

$$A' \cup B' = \{x : x \notin A \text{ or } x \notin B\}.$$

Proof:

We want to show that:

$$(A \cap B)' = A' \cup B'.$$

Logical form of the theorem:

$\sim(p \wedge q) = \sim p \vee \sim q$

p	q	~p	~q	$(p \wedge q)$	$\sim(p \wedge q)$	$\sim p \vee \sim q$
T	T	F	F	T	F	F
T	F	F	T	F	T	T
F	T	T	F	F	T	T
F	F	T	T	F	T	T

From the last two columns of the truth table we observe that $\sim(p \wedge q) = \sim p \vee \sim q$

Hence, $(A \cap B)' = A' \cup B'$.

14. If x and y are positive real numbers and $x^2 < y^2$ then $x < y$

Solution: We are tasked with proving that if x and y are positive real numbers and $x^2 < y^2$, then $x < y$.

Given:

- x and y are positive real numbers.
- $x^2 < y^2$.

To Prove:

- $x < y$.

Proof:

Since x and y are positive real numbers, we can take the square root of both sides of the inequality $x^2 < y^2$. Here's how:

- From the assumption, we know that $x^2 < y^2$.
- Since x and y are positive, we can take the square root of both sides without changing the direction of the inequality:

$$\sqrt{x^2} < \sqrt{y^2}.$$

- The square root of x^2 is x (because x is positive), and the square root of y^2 is y (because y is also positive):

$x < y$.

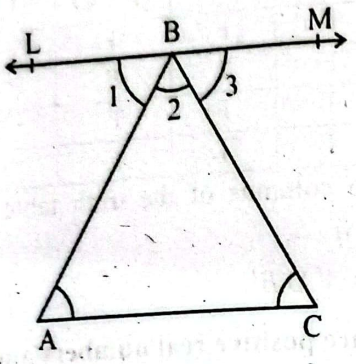
Thus, if $x^2 < y^2$, then $x < y$ for positive real numbers x and y.

Q.E.D.

15. The sum of the interior angle of a triangle is 180^0

Ans. Triangle is 180^0

Given: A ΔABC



To prove

$$m\angle A + m\angle B + m\angle C = 180^o$$

Construction:

Passing through vertex B, draw $\overleftrightarrow{LM} \parallel \overline{AC}$

Proof:

$m\angle 1 + m\angle 2 + m\angle 3 = 180^o$..... (i)	Sum of st. Line angles
$m\angle 1 + m\angle A$..... (ii)	Alternate angles are equal.
$m\angle 2 = m\angle ABC$..... (iii)	From figure
$m\angle 3 = m\angle C$..... (iv)	Alternate angles are equal
Now (i) can be written as	
$m\angle A + m\angle ABC + m\angle C = 180^o$	From (i), (ii), (iii) and (iv)
$m\angle A + m\angle B + m\angle C = 180^o$	$m\angle ABC = m\angle B$

Hence, the sum of the interior angles of a triangle is 180^0.

16. If a, b and c are non-zero real numbers, prove that:

(i) $\frac{a}{b} = \frac{c}{d} \Leftrightarrow ad = bc$ (ii) $\frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd}$

(iii) $\frac{a}{b} + \frac{c}{b} = \frac{a+c}{b}$.

Sol: Let's prove each of the statements:

(i) $\frac{a}{b} = \frac{c}{d} \Leftrightarrow ad = bc$

Proof:

Given $\frac{a}{b} = \frac{c}{d}$, we can write:

$$a \cdot d = b \cdot c \quad \text{(cross-multiplication).}$$

Thus, $\frac{a}{b} = \frac{c}{d} \Rightarrow ad = bc$.

Conversely, if $ad = bc$, divide both sides by $b \cdot d$ (since b and d are non-zero): $\frac{a}{b} = \frac{c}{d}$.

Hence, $\frac{a}{b} = \frac{c}{d} \Rightarrow ad = bc$.

(ii) $\frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd}$

Proof: Using the definition of fractions:

$$\frac{a}{b} \cdot \frac{c}{d} = \frac{a \cdot c}{b \cdot d}.$$

Hence, the product of two fractions is $\frac{ac}{bd}$, as required.

(iii) $\frac{a}{b} + \frac{c}{b} = \frac{a+c}{b}$

Proof: Since the denominators are the same (b), we add the numerators directly: $\frac{a}{b} + \frac{c}{b} = \frac{a+c}{b}$.

Thus, the sum of the fractions is $\frac{a+c}{b}$.

Conclusion:

All three statements have been proven.